Transpose matrices

A $n\times m$ matrix. The transpose A^T is an $m\times n$ matrix with

$(A^T)_{ij} = (A)_{ji}$

Example $A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 2 & 0 \\ 1 & 0 \\ 4 & 3 \end{bmatrix}$

useful facts • $(A^T)^T = A$

• $(\text{upper triangular})^T = (\text{lower triangular})$

• $(A+B)^T = A^T + B^T$

• $(AB)^T = B^T A^T$

• $(A^{-1})^T = (A^T)^{-1}$ ← check:

$$AA^{-1} = I$$
$$(AA^{-1})^T = I^T = I$$
$$\parallel$$
$$(A^{-1})^T A^T = I \Rightarrow (A^{-1})^T \text{ is inverse for } A^T.$$

Symmetric matrices

A matrix is <u>symmetric</u> if $A^T = A$

Examples I $\quad [3]$ $\quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$

Fact if A is symmetric, and A^{-1} exists, then A^{-1} is symmetric.

Proof A symmetric $\Rightarrow A^T = A$ so $(A^T)^{-1} = A^{-1}$

$$\parallel$$
$$(A^{-1})^T \quad \text{so } (A^{-1})^T = A^{-1} \quad \square.$$

Symmetric products

Fact: let R be an $(m \times n)$ matrix then $R^T R$ is a square symmetric matrix.

Proof $\underset{n\times m}{R^T}\ \underset{m \times n}{R} = n \times n.$ $\qquad (R^TR)^T = R^T (R^T)^T = R^T R$

$\Rightarrow$ symmetric $\square$.

Example $\begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = [5]$ $\qquad \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Fact suppose A is symmetric $(A = A^T)$ and $A = LDU$ with no row swaps then $U = L^T$, so $A = LDL^T$. This is the symmetric factorization

Proof $A = LDU$

$A^T = U^T D^T L^T = U^T D L^T$, but U, L unique $\Rightarrow L = U^T$ $\square$.

Special matrices and applications

example solve $-\dfrac{d^2u}{dx^2} = f(x)$ $\qquad 0 \le x \le 1$

solution: a function $u(x)$

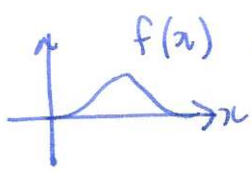
note: if $u(x)$ is a solution $u(x) + C + Dx$ is also a solution.

e.g. if $f(x) = 0$, then $u(x) = C + Dx$ is a solution.

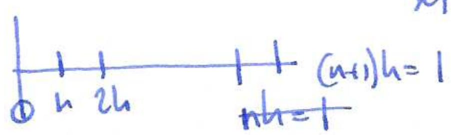
boundary conditions $u(0) = 0$, $u(1) = 0$ $\rightsquigarrow$ determines C, D.

application: temperature distribution in a rod with heat source $f(x)$.

aim: find approximate solution by discretization.



$\leftarrow$ infinitely many values of x



set $u_i = f(ih)$

$u_0 = 0$

$u_{n+1} = 0$.

approximate derivative $\qquad \dfrac{\Delta u}{\Delta x} = \dfrac{u(x+h) - u(x)}{h}$

symmetric version : $\quad \dfrac{du}{dx} \approx \dfrac{\Delta u}{\Delta x} = \dfrac{u(x+h) - u(x-h)}{2h}$

approx 2nd derivative $\qquad \dfrac{d^2u}{dx^2} = \dfrac{u(x+h) - 2u(x) + u(x-h)}{h^2}$

$-\dfrac{d^2u}{dx^2} = f(x) \quad : \quad - \dfrac{u_{j+1} - 2u_j - u_{j-1}}{h^2} = f(jh)$

Example $\quad h = \frac{1}{6}$

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_5 \end{bmatrix} = h^2 \begin{bmatrix} f(h) \\ f(2h) \\ \vdots \\ f(5h) \end{bmatrix}$$

properties :
- tridiagonal (sparse)
- symmetric $\quad A = A^T \qquad A = LDL^T$
- positive definite $\Rightarrow$ all pivots are positive.

Elimination

$② - \frac{1}{2}①$

$$\begin{bmatrix} 2 & -1 & & & \\ 0 & 3/2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & 1 & 2 \end{bmatrix} \quad \text{etc.}$$

obtain
$$A = \underbrace{\begin{bmatrix} 1 & & & & \\ -1/2 & 1 & & & \\ & -2/3 & 1 & & \\ & & -3/4 & 1 & \\ & & & -4/5 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2/1 & & & & \\ & 3/2 & & & \\ & & 4/3 & & \\ & & & 5/4 & \\ & & & & 6/5 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & -1/2 & & & \\ & 1 & -2/3 & & \\ & & 1 & -3/4 & \\ & & & 1 & -4/5 \\ & & & & 1 \end{bmatrix}}_{L^T}$$

$\det(A) = \det(D) = \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdot\frac{6}{5} = 6$. complexity $O(n)$.

Round off error

Example $A = \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix}$ ill conditioned $B = \begin{bmatrix} 0.0001 & 1 \\ 1 & 1 \end{bmatrix}$ well conditioned

A: $u + v = 2$ $u + v = 2$

$u + 1.0001v = 2$ $u + 1.0001v = 2.0001$

solution $u = 2$, $v = 0$ $u = 1$, $v = 1$

naive elimination on B $\left[\begin{array}{cc|c} 0.0001 & 1 & 1 \\ 1 & 1 & 2 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 0.0001 & 1 & 1 \\ 0 & -9999 & -9998 \end{array}\right]$

round off in back substitution, get $v \approx 1$, $u = 0$

correct result: $v = 0.9999$ $u = 1$.

moral: small pivots bad!

solution: for best numerical results swap rows to get largest pivot

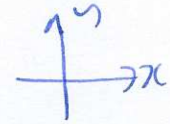
this is called: elimination with partial pivoting.

Vector spaces and subspaces

Examples $\mathbb{R}^n$

$\mathbb{R}$: numbers.

$\mathbb{R}^2$: $\langle x, y \rangle$



$\mathbb{R}^3$: $\langle x, y, z \rangle$

$\mathbb{R}^n$: $\langle x_1, x_2, x_3, \ldots, x_n \rangle$

key properties: linear combinations, i.e. we can add vectors and take scalar multiples.

Properties V vector space $x, y, z \in V$, $c_1, c_2 \in \mathbb{R}$.

addition: $x + y = y + x$ commutes

$x + (y+z) = (x+y) + z$ associative

$0 + x = x + 0 = x$ zero vector

$x + (-x) = 0$ inverse (additive)

scalar multiplication: $1x = x$ identity

$(c_1 c_2)x = c_1(c_2 x)$

$c(x+y) = cx + cy$ distribution

$(c_1 + c_2)x = c_1 x + c_2 x$ distribution.

Examples · $\mathbb{R}^\infty$ $(x_1, x_2, \ldots)$ $(0, 0, 0, \ldots)$
$(1, 1, 1, \ldots)$

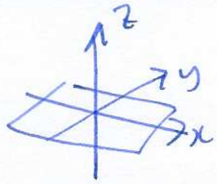
· collections of 3x2 matrices. $\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$ $A+B$ $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

space of functions $f(x)$ on $[0,1]$ $f(x) + g(x)$ $cf(x)$ zero: $f(x) = 0$

subspaces

example

xy-plane $\subseteq \mathbb{R}^3$.



Def^n A subspace of a vector space is a non-empty subset that satisfies the requirements for a vector space: linear combinations stay in the subspace. i.e. if

$x, y \in$ subspace then $x+y \in$ subspace

$x \in$ subspace then cx in subspace.

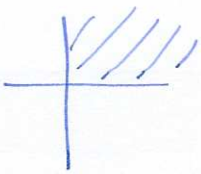
Notation we say the subspace is closed under addition and scalar multiplication

Observation $0.x = 0$ vector, so every subspace contains the zero vector.

- the smallest subspace is $\{0\}$ zero vector
- the largest subspace of V is all of V.

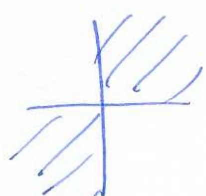
Examples

① all vectors in $\mathbb{R}^2$ with non-negative components $\langle x, y \rangle$ $x \geq 0$, $y \geq 0$



not a subspace (satisfies addition but not scalar mult)

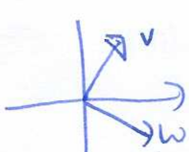
② 1st and 3rd quadrants



(satisfies scalar mult, but not addition).

③ $V = 3 \times 3$ matrices subspaces: lower triangular matrices, symmetric matrices.

④



$A = \{\lambda v \mid \lambda \in \mathbb{R}\}$ is a subspace.

$B = \{\lambda w \mid \lambda \in \mathbb{R}\}$ "

$A \cup B$ not a subspace in general.