**Remark** a linear combination of vectors $v_1, v_2 \ldots v_n$ is a sum $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$, $a_n \in \mathbb{R}$.

## Gaussian elimination, general case

**Example** $\begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ swap rows: $\begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \cancel{y}\,x \\ y \end{bmatrix} = \begin{bmatrix} b_2 \\ b_1 \end{bmatrix}$

**Observation** we can swap rows by multiplying by a matrix, called a permutation matrix. In this case $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ check $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}$

original system : $Ax = b$
new system : $PAx = Pb$

**Defn** A permutation matrix has the same rows as the identity, in some order, so there is a single 1 in each row and column.

**Example** $P = I_n$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

**Q**: how many permutation matrices? **A**: $n!$

**useful facts** • the product of two permutation matrices is a permutation matrix

• $P^{-1} = P^T$, where $(P^T)_{ij} = (P)_{ji}$ ("turn $P$ on its side")

**Example** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**Fact** (1.5 1J) In the non-singular case ($n \times n$, unique solution) there is a permutation matrix $P$ that reorders the rows so we can do Gaussian elimination with elementary matrices / row operations w/ no swaps then $PAx = b$ has a unique solution and $PA = LU$ or $LDU$

(singular case more complicated...) <u>example</u> $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

## §1.6 Inverses and transposes

Let $A$ be an $n \times n$ matrix. The inverse of $A$ is an $n \times n$ matrix $A^{-1}$ s.t. $A^{-1}A = I_n$ and $AA^{-1} = I_n$

<u>warning</u>: not all $n \times n$ matrices have inverses.

<u>examples</u> $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

<u>Observations</u>

① The inverse exists iff elimination produces $n$ pivots (i.e. unique solution)

② left inverse = right inverse

<u>Proof</u> suppose $BA = I$, $AC = I$ then $(BA)C = B(AC)$, $IC = BI \Rightarrow C = B$. □.

③ ~~above shows~~ inverse is unique. <u>Proof</u> $BA = I$, $CA = I$ $(BA)C = B(AC)$, $IC = BI$, $C = B$. □.

④ If $A$ is invertible, then $Ax = b$ has the unique solution $x = A^{-1}b$

$Ax = b$ $\quad \underbrace{A^{-1}A}_{I}x = A^{-1}b$ $\quad Ix = A^{-1}b$ $\quad x = A^{-1}b$.

⑤ suppose there is a <u>non-zero</u> vector $x$ s.t. $Ax = 0$. then $A$ <u>cannot</u> have an inverse. as $\underbrace{A^{-1}A}_{I}x = A^{-1}0 = 0 \Rightarrow Ix = 0 \Rightarrow x = 0$ ※.

i.e. if $A$ has an inverse, then $Ax = 0$ has <u>exactly one</u> solution, $x = 0$.

⑥ $2 \times 2$ matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff $\underbrace{ad - bc}_{\det(A)} \neq 0$

if $ad - bc \neq 0$ then $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

check!
$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

⑦ a diagonal matrix has an inverse iff no diagonal elements are zero

$$\begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix}^{-1} = \begin{bmatrix} 1/d_1 & & & 0 \\ & 1/d_2 & & \\ & & \ddots & \\ 0 & & & 1/d_n \end{bmatrix}$$

Propⁿ: (1.6 1L) If $A$ has inverse $A^{-1}$ and $B$ $B^{-1}$ then $(AB)^{-1} = B^{-1}A^{-1}$

Corollary $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

Q: how do we find $A^{-1}$? A: Gaussian elimination / row operations!

idea $AA^{-1} = I$ think of this as an equation, and solve for $A^{-1}$

$$A\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = I \qquad \begin{bmatrix} Ax_1 & Ax_2 & \cdots & Ax_n \end{bmatrix} = I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & & \vdots \\ \vdots & \vdots & & 0 \\ 0 & 0 & & 1 \end{bmatrix}$$

could solve $Ax_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $Ax_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, ..., $Ax_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ separately

better: do them all at the same time!

Example $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$ solve $Ax_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = e_1$ etc. $\begin{bmatrix} 2 & 1 & 1 & 1 \\ 4 & -6 & 0 & 0 \\ -2 & 7 & 2 & 0 \end{bmatrix}$

do elimination / row reduction on $\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} A & I \end{bmatrix}$

row operations: $\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} U & L^{-1} \end{bmatrix}$$

now clear entries <u>above</u> the pivots.

$$\begin{bmatrix} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 & 12/8 & -5/8 & -6/8 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

<u>Fact</u> $\det(A) =$ product of pivots $= -16$.

divide by pivots.

$$\begin{bmatrix} 1 & 0 & 0 & 12/16 & -5/16 & -6/16 \\ 0 & 1 & 0 & 1/2 & -3/8 & -1/4 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix} \qquad \text{check!}$$

<u>Fact</u> this method of computing $A^{-1}$ takes $\leq \frac{4}{3} n^3$ operations.
in fact $n^3$.

<u>Thm</u> A $n \times n$ matrix is invertible $\Leftrightarrow$ A has a full set of $n$ pivots when you do row reduction

$\Leftarrow$ suppose A has a full set of pivots, then solving $AA^{-1} = I$ equivalent to solving $Ax_i = e_i$, unique solution $\Rightarrow A^{-1} = [x_1 \, x_2 \cdots x_n]$ is inverse.

<u>Recall</u> a 1-sided inverse is automatically a 2-sided inverse.

$\Rightarrow$ suppose A does not have $n$-pivots. then can row reduce to

$$\begin{bmatrix} d_1 & * & * & * \\ 0 & d_2 & * & * \\ 0 & 0 & 0 & d_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\leftarrow$ get whole row of zeros, so there is a non-zero vector $x$ s.t. $Ax = 0$. $\square$.