special matrix : identity matrix $I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

$I_1 = [1]$ $\quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdots$

key property $Ix = x$

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Matrix multiplication $\quad \underbrace{A}_{m \times n} \underbrace{B}_{n \times p} = \underbrace{AB}_{m \times p}$

rows x cols $\quad$ rows x cols

these must be equal!

Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 2 & 11 \end{bmatrix}$$

$$AB = \begin{bmatrix} (AB)_{11} & (AB)_{12} \cdots \\ & \end{bmatrix}$$

$(AB)_{11}$ = 1st row of $A$ · 1st col of $B$

$= a_{11} b_{11} + a_{12} b_{21} + \cdots + a_{13} b_{31}$

$= \sum_{j=1}^{n} a_{ij} b_{j1}$

$(AB)_{ij}$ = ($i$-th row of $A$) · ($j$-th col of $B$)

$$= \sum_{k=1}^{n} a_{ik} b_{kj}$$

column view $\quad \underset{m \times n}{A} \underset{n \times p}{B} \quad$ $B$ has columns $[b_1 \; b_2 \; \ldots \; b_p]$

$$AB = [Ab_1 \; Ab_2 \cdots \; Ab_p]$$

Q: why do we use this def$^{n}$ of matrix multiplication?

A: because it is useful / corresponds to linear maps.

<u>observation</u> : Gaussian elimination operations are given by matrix multiplication by special matrices called <u>elementary matrices</u>

<u>recall</u> :

$$\begin{aligned} 2x + y + z &= 5 \quad (1) \\ 4x - 6y &= -2 \quad (2) \\ -2x + 7y + 2z &= 9 \quad (3) \end{aligned} \qquad \left[\begin{array}{cccc} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{array}\right]$$

step (1) : (1), (2) − 2(1), (3)

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ -2 & 7 & 2 & 9 \end{bmatrix}$$

step (2) : (1), (2), (3) + (1)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ -2 & 7 & 2 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{bmatrix}$$

etc.

can also swap rows

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \\ a_{21} & a_{22} \\ a_{41} & a_{44} \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

swap 2,3. — swap 1 and 3 rows.

<u>Properties of matrix multiplication</u>

$$\underbrace{A}_{m\times n}\ \underbrace{B}_{n\times p} = \underbrace{AB}_{m\times p}$$

- each entry $(AB)_{ij}$ is the product of the i-th row of A and the j-th col of B
- the j-th col of AB is the product of A by the j-th col of B.
- the i-th row of AB is the product of the ith row of A by B.

Associativity $(AB)C = A(BC)$ so can just write $ABC$

Distributive $A(B+C) = AB + AC$

$(A+B)C = AC + BC$

not commutative! $AB \neq BA$

note: $\underset{m\times n}{A} \times \underset{n\times p}{B}$ makes sense $\underset{n\times p}{B}\ \underset{m\times n}{A}$ doesn't work!

but even if $A, B$ $n\times n$, $AB \neq BA$ in general

Example
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

## §1.5 Triangular factors and row exchanges

back to elimination example $Ax = b$
$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

$$\begin{array}{l} ① \\ ② - 2① \\ ③ \end{array} \underset{E}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix}} = \underset{EA}{\begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ -2 & 7 & 2 & 9 \end{bmatrix}}$$

$$\begin{array}{l} ① \\ ② \\ ③ + ① \end{array} \underset{F}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}} \underset{EA}{\begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ -2 & 7 & 2 & 9 \end{bmatrix}} = \underset{FEA}{\begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{bmatrix}}$$

$$\begin{array}{l} ① \\ ② \\ ③ + ② \end{array} \underset{G}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}} \underset{FEA}{\begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{bmatrix}} = \underset{GFEA}{\begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{bmatrix}}$$

upper triangular, can solve by back substitution

$Ax = b$

$$\underbrace{GFE}_{\text{elementary matrices}} A = \underbrace{U}_{\text{upper triangular}}$$

key property ① : a product of upper triangular matrices is upper triangular
" lower " lower "

$\underbrace{GFE}$ ← lower triangular!

key property ② each row operation is reversible, so each elementary matrix has an inverse

Defn an inverse for an $n\times n$ matrix $A$ is an $n\times n$ matrix, $A^{-1}$ say, s.t. $A^{-1}A = I_n$ ← identity matrix.

warning: not all matrices have inverses! (e.g. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$).

Example $E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

check: $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

note: $GFEA = U$ $\qquad FEA = G^{-1}U$ $\qquad EA = F^{-1}G^{-1}U$

$\underbrace{G^{-1}G}_{I_3} FEA = G^{-1}U$ $\qquad \underbrace{F^{-1}F}_{I_3}EA = F^{-1}G^{-1}U$ $\qquad \underbrace{E^{-1}E}_{I_3}A = \underbrace{E^{-1}F^{-1}G^{-1}}_{\text{lower triangular}}\underbrace{U}_{\text{upper triangular}}$

$A = \underbrace{L}_{\text{lower triangular, 1's on diagonal}}\underbrace{U}_{\text{upper triangular}}$

Thm Every matrix $A$ has an $LU$ factorization, where
$U$ is upper triangular, result of doing Gaussian elimination
$L$ is lower triangular, 1's on the diagonal, corresponds to the Gaussian elimination row operations.

Thm (1.5 1H) If $A$ can be reduced with no exchange of rows, then $A = LU$

Q: why do this?

A: suppose we want to solve $\underbrace{Ax = b}_{\text{linear system}}$ $\longleftrightarrow$ $\underbrace{LUx = b}_{\text{two triangular systems}}$

$O(n^3)$

solve $Lc = b$ $O(n^2)$

then $Ux = c$ $O(n^2)$

we often want to solve $Ax = b$ for many different values of $b$

so we can do this by ① factor $A = LU$ ($O(n^3)$ operations)

② solve $Lc = b$, $Ux = c$ $O(n^2)$ operations.

Example solve $Ax = b$

$$\begin{aligned} x + 2y - z &= b_1 \\ -x \quad + 3z &= b_2 \\ x + 6y + 6z &= b_3 \end{aligned} \qquad A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 0 & 3 \\ 1 & 6 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 4 & 7 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \circledast$$

$b = \begin{bmatrix} +1 \\ 1 \\ -1 \end{bmatrix}$ solve $Lc = b$ $\rightarrow$ $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ $c = \begin{bmatrix} 1 \\ 2 \\ -6 \end{bmatrix}$

$Ux = c$

$Ux = c \rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -6 \end{bmatrix}$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -14 \\ 3 \\ -2 \end{bmatrix}$.

symmetric version of factorization: $A = LDU$ ← upper triangular, 1s on diagonal

lower triangular, 1s on diagonal

diagonal (may have zeros on diagonal)

Example ⊛ $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Fact if $A = LU$, then $L, U$ unique

if $A = LDU$, $L, D, U$ unique.