in $\mathbb{R}^n$: also have inner products / angles / lengths.

inner products : $\underline{v},\underline{w}$: $\underline{v}\cdot\underline{w} = \|v\|\,\|w\|\cos\theta$ (geometric)

$= \langle v_1,\dots,v_n\rangle\cdot\langle w_1,\dots,w_n\rangle$

$= v_1w_1 + v_2w_2 + \dots + v_nw_n$ (coords).

Thm these two defs are equivalent. $\square$.

properties: • $\underline{v}\cdot\underline{v} = \|v\|\,\|v\|\cos(0) = \|v\|^2$ ← can use inner product to find length of vector.

• to find angle: $\underline{v}\cdot\underline{w} = \|v\|\,\|w\|\cos\theta \Rightarrow \cos\theta = \dfrac{\underline{v}\cdot\underline{w}}{\|v\|\,\|w\|}$ $\quad\theta = \cos^{-1}\left(\dfrac{\underline{v}\cdot\underline{w}}{\|v\|\,\|w\|}\right)$.

• two vectors are perpendicular if $\underline{v}\cdot\underline{w} = 0$.

• unit vectors: $\hat{\underline{v}}$ = unit vector in direction of $\underline{v}$. $\hat{v} = \underline{v}\cdot\dfrac{1}{\|v\|}$.

Examples: $\underline{v}$ $\langle 1,2,3\rangle$ ← $\hat{v} = \langle 1,2,3\rangle\cdot\dfrac{1}{\sqrt{1^2+2^2+3^2}} = \dfrac{1}{\sqrt{12}}\langle 1,2,3\rangle$.

angle between $\langle 1,2,3\rangle$, $\langle 1,0,-1\rangle$ is

$$\frac{\langle 1,2,3\rangle\cdot\langle 1,0,-1\rangle}{\|\langle 1,2,3\rangle\|\,\|\langle 1,0,-1\rangle\|} = \frac{1-3}{\underbrace{\sqrt{12}\cdot\sqrt{2}}_{\sqrt{2}\sqrt{6}}} = \frac{-2}{2\sqrt{6}} = -\frac{1}{\sqrt{6}}$$

$\theta = \cos^{-1}(-1/\sqrt{6})$.

Recall $\quad x+2y=3$, $\quad 4x+5y=6$ ← can think of ① 2 lines in $\mathbb{R}^2$.

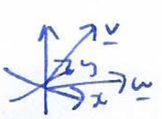
② $\begin{bmatrix} x \\ 4x \end{bmatrix} + \begin{bmatrix} 2y \\ 5y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$.

a collection of linear equation ⟷ a single vector equation $\begin{bmatrix} 1 \\ 4 \end{bmatrix}x + \begin{bmatrix} 2 \\ 5 \end{bmatrix}y = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$.

⤢

Q: can you make the vector $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ as a sum of the vectors $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$?

Q: what do sums of vectors look like?



$\underline{v}x$ ⟷ line of in direction $\underline{v}$.

$\underline{v}x + \underline{w}y$ ← plane containing $\underline{v}$ and $\underline{w}$.

Gaussian Elimination

$$2x + y + z = 5 \quad (1)$$
$$4x - 6y = -2 \quad (2)$$
$$-2x + 7y + 2z = 9 \quad (3)$$

notation: pivot ·

(1) $\boxed{2}x + y + z = 5$

$(2) - 2(1)$ $\boxed{-8}y - 2z = -12$

$(3) + (1)$ $8y + 3z = 14$

$$\left.\begin{aligned} (1) \quad & 2x + y + z = 5 \\ (2) \quad & -8y - 2z = -12 \\ (3) + (2) \quad & z = 2 \end{aligned}\right\}$$

triangular system, can solve by back substitution

$z = 2$

$-8y - 4 = -12 \Rightarrow y = 1$

$2x + 1 + 2 = 5 \Rightarrow x = 1$

more efficient notation:

$$\begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{bmatrix} \to \begin{bmatrix} \boxed{2} & 1 & 1 & 5 \\ 0 & \boxed{-8} & -2 & -12 \\ 0 & 0 & \boxed{1} & 2 \end{bmatrix}$$

pivots

important: pivots can not be zero.

Q: what can go wrong? A: a zero can appear in a pivot position.

Example (avoidable)

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 6 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 2 & 4 \end{bmatrix} \xrightarrow{\text{swap!}} \begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{2} & 4 \\ 0 & 0 & \boxed{3} \end{bmatrix}$$

Example (unavoidable)

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 4 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & 0 & \boxed{3} \\ 0 & 0 & 0 \end{bmatrix}$$

only two pivots.

Q: how many operations do we need for elimination?

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ \vdots & & & \\ a_{n1} & & & a_{nn} \end{bmatrix} \rightsquigarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & & & \\ \vdots & & \boxed{} & \\ 0 & & & \end{bmatrix}$$

$n \times n$ matrix

need $(n-1) \times n$ operations.

$(n-1) \times (n-1)$ matrix

so need $(n-1)n + (n-2)(n-1) + \cdots$ operations.

$$\begin{aligned} &= n^2 - n + (n-1)^2 - (n-1) + \cdots + 1^2 - 1 \\ &= 1 + 2^2 + 3^2 + \cdots + n^2 - (1 + 2 + 3 + \cdots + n) \\ &\quad \tfrac{1}{6} n(n+1)(2n+1) - \tfrac{1}{2} n(n+1) = \tfrac{1}{3}(n^3 - n) \sim \tfrac{1}{3} n^3 . \end{aligned}$$

§1.4 Matrix notation and matrix multiplication

Example

$$\begin{aligned} 2x + y + z &= 5 \\ 4x - 6y &= -2 \\ -2x + 7y + 2z &= 9 \end{aligned}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

coefficient matrix A — vector of unknowns x — target vector b

notation $Ax = b$

Defn A matrix is a rectangular array of numbers. $m \times n$ matrix, rows $\times$ cols.

Examples

$[3]$ 1×1 $\quad$ $[2 \;\; 1]$ 1×2 $\quad$ $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ 3×1 $\quad$ $\begin{bmatrix} 2 & 3 & 4 \\ 1 & 1 & 1 \end{bmatrix}$ 2×3.

operations: if two matrices have the same size, can add them

Example
$$\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 4 & 3 \end{bmatrix}$$

$$[1\ 0\ 1] + [-1\ -1\ -1] = [0, -1, 0]$$

scalar multiplication can multiply a matrix by a number

Examples
$$2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

matrix multiplication

special case
$$\underbrace{[1\ 2\ 3]}_{1\times 3}\underbrace{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}}_{3\times 1} = [1\cdot 4 + 2\cdot 5 + 3\cdot 6] = \underbrace{[32]}_{1\times 1}$$

↑ same def^{n} as inner product!

more generally (2 equivalent views)

$$\begin{bmatrix} 1 & 1 & 6 \\ 3 & 0 & 1 \\ 1 & 1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1\cdot 2 + 1\cdot 5 + 6\cdot 0 \\ 3\cdot 2 + 0\cdot 5 + 1\cdot 5 \\ 1\cdot 2 + 1\cdot 5 + 4\cdot 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 7 \end{bmatrix}$$

take inner product of each row of first matrix with each column of second matrix

equivalently: Ax is $2\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 6 \\ 1 \\ 4 \end{bmatrix}$

a linear combination of the columns of A

more generally
$$\underbrace{A}_{m\times n}\ \underbrace{x}_{n\times 1} = \underbrace{Ax}_{m\times 1} \qquad \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x}$$

$$Ax = \begin{bmatrix} (Ax)_1 \\ \vdots \\ (Ax)_m \end{bmatrix} \qquad (Ax)_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \overset{\triangle}{=} \sum_{j=1}^{n} a_{ij}x_j$$

$$\vdots$$

$$(Ax)_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n =$$