Math 338 Linear Algebra Spring 22 Midterm 3a

Name: Solutions

- Do any 8 of the following 10 questions.
- You may use a calculator without symbolic algebra capabilities, but no notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 3	
Overall	

(1) Let A be the matrix $A = \begin{bmatrix} 1 & 6 \\ -1 & -4 \end{bmatrix}$.

(a) Find the eigenvalues of A.

(b) What are the eigenvalues for A^k? Explain your answer.

a) $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 6 \\ -1 & -4-\lambda \end{vmatrix} = (1-\lambda)(-4-\lambda) + 6 = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2)$

$\lambda = -1, -2$

b) if $Av = \lambda v$ then $A(Av) = \lambda(Av)$.

so $A^2 v = \lambda^2 v$ etc.

so eigenvalues for A^k are $(-1)^k, (-2)^k$.

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "A"] & \mathbb{R}^2 \\ \mathbb{R}^2 \arrow[u, "P"] \arrow[r, "D"'] & \mathbb{R}^2 \arrow[u, "P"'] \end{tikzcd}$$

(2) Let A be the same matrix as in Q1, i.e. $A = \begin{bmatrix} 1 & 6 \\ -1 & -4 \end{bmatrix}$.

(a) Find the eigenvectors for A.

(b) Diagonalize A, i.e. find matrices P and D such that $P^{-1}AP = D$.

a) $\lambda = -1$ solve $(A+I)x = 0$: $\begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$ $x = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$\lambda = -2$ solve $(A+2I)x = 0$: $\begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ $x = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$D = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ $P = \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix}$

check:

$$\begin{bmatrix} +1 & +2 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} \cancel{4} & \cancel{6} \\ \cancel{6} & \cancel{2} \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix} = \cancel{\begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}} \cancel{\begin{bmatrix} -3 & -2 \\ 2 & 2 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 6 \\ -1 & -4 \end{bmatrix} \qquad = \cancel{[\ 1}$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} -3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \checkmark$$

(3) Let A be the same matrix as in Q1, i.e. $A = \begin{bmatrix} 1 & 6 \\ -1 & -4 \end{bmatrix}$.

(a) Write down a product of matrices which gives A^k.
(b) Write down a product of matrices which gives e^{At}.
(c) What can you say about e^{At} as $t \to \infty$?

a) $D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ $P = \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix}$ $A = PDP^{-1}$

$A^k = PD^kP^{-1}$

b) $e^{At} = I + At + A^2\frac{t^2}{2!} + \dots$

$= P\left(I + Dt + D^2\frac{t^2}{2!} + \dots\right)P^{-1} = P\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}P^{-1}$

c) $e^{At} \to 0$ as $t \to \infty$.

(4) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for $\mathbb{R}^3$, where

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}.$$

Use the Gram-Schmidt process to find an orthonormal basis.

$$q_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \qquad q_2' = v_2 - (v_2 \cdot q_1)\, q_1$$

$$q_2' = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} - \frac{1}{\sqrt{3}} \cdot 3\, \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$q_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$q_3' = v_3 - (q_1 \cdot v_3)\, q_1 - (q_2 \cdot v_3)\, q_2 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} - \frac{1}{\sqrt{3}}\, 4\, \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - \underbrace{\frac{1}{\sqrt{2}} \cdot 0 \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}}_{0.}$$

$$= \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} - \frac{4}{3}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$q_3 = \frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

(5) (a) Suppose A is an $n \times n$ matrix and $A^2 = A$. What can you say about $\det(A)$?
(b) Suppose A is an $n \times n$ matrix and $\det(A) = 0$. What can you say about the eigenvalues of A?

a) $A^2 = A \Rightarrow \det(A)^2 = \det(A) \Rightarrow \det(A)^2 - \det(A) = 0$

$\det(A)(\det(A) - 1) = 0$

$\det(A) = 0, 1$

b) at least one eigenvalue is 0

$$\begin{array}{c} \mathbb{R}^2_E \\ \uparrow P \\ \mathbb{R}^2_B \end{array} \qquad P = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

(6) Let B be the basis for $\mathbb{R}^2$ given by

$$B = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}.$$

(a) Find a matrix which converts vectors written in the standard basis to vectors written with respect to the basis B.

(b) Use your answer to (a) to write $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (in the standard basis) as a linear combination of vectors in B.

a) $P^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

b) $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}_E = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_B.$

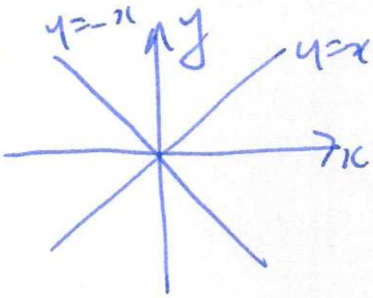
(7) (a) Write down a matrix A corresponding to an anticlockwise rotation of $\pi/4$ about the origin in $\mathbb{R}^2$.

(b) Write down a matrix B which expands $\mathbb{R}^2$ by a factor of 2 in the x-direction, and also reflects across the x-direction.

(c) Use your answers above to find a matrix which expands $\mathbb{R}^2$ by a factor of 2 in the line $y = x$, and a ~~factor of 3 in the line $y = -x$~~.
reflects across $y = x$

a) $$\begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

b) $$\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

c) 

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 3/\sqrt{2} \\ 3/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

(8) Let $A = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}$.

(a) Find the eigenvalues and eigenvectors for A.

(b) Can you diagonalize A? Explain.

a) $\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 4 \\ -1 & 3-\lambda \end{vmatrix} = (-1-\lambda)(3-\lambda) + 4$

$= \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$

$\lambda = 1, 1.$

eigenvectors, solve $(A - I)x = 0$

$\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \sim \begin{bmatrix} -2 & 4 \\ 0 & 0 \end{bmatrix} \quad x = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

only one eigenvector, so can't diagonalize.

(9) Let $J = \begin{bmatrix} 0 & x & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix}$, where x and y may be either 0 or 1.

(a) What are the eigenvalues of A?
(b) What are the largest and smallest number of eigenvectors that A may have?
(c) Suppose $A = PJP^{-1}$, for some invertible matrix P. Show that $A^3 = 0$.

a) $\lambda = 0, 0, 0$

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow$ 1 eigenvalue

b) # eigenvectors ≥ 1, ≤ 3.

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow$ 2 eigenvalues $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ 3 eigenvalues

c) $J^2 = \begin{bmatrix} 0 & 0 & xy \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad J^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$

so $A^3 = P \cdot 0 \cdot P^{-1} = 0$.

(10) Let A be a matrix with eigenvalues $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 2$ and $\lambda_4 = -2$, and the following orthonormal eigenvectors

$$v_1 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, v_2 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, v_3 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, v_4 = \frac{1}{2}\begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Write the vector $b = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ with respect to the basis of eigenvectors. (Hint: use the fact that the v_i are orthogonal.)

(b) Use your answer above to find Ab with respect to the basis of eigenvectors.

a) $b = \frac{3}{2} v_1 + \frac{3}{2} v_2 + -\frac{1}{2} v_3 + \frac{1}{2} v_4 = \begin{bmatrix} 3/2 \\ 3/2 \\ -1/2 \\ 1/2 \end{bmatrix}_B.$

b) $Ab = 1 \frac{3}{2} v_1 + (-1)\frac{3}{2} v_2 + 2 \cdot \left(-\frac{1}{2}\right) v_3 - 2 \frac{1}{2} v_4.$

$= \begin{bmatrix} 3/2 \\ -3/2 \\ -1 \\ -1 \end{bmatrix}_B.$