Linear Algebra Spring 22 Sample Midterm 3

(1) Let A be the matrix $A = \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix}$.

(a) Find the eigenvalues of A.
(b) Diagonalize A, i.e. find matrices P and D such that $P^{-1}AP = D$.
(c) Find an exact formula for A^k.
(d) What are the eigenvalues for A^k?

(2) Let $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $L(x, y) = (2x - y, 2x + y)$. Let

$$S = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

be a basis for $\mathbb{R}^2$, and let T be the standard basis for $\mathbb{R}^2$.

(a) Find the matrix for L with respect to T.
(b) Find the matrix for L with respect to S.
(c) What is the rank and nullity of L?

(3) Suppose q_1, q_2 and q_3 are orthonormal vectors in $\mathbb{R}^3$ such that $\det \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} = 1$. Find the values of the following determinants, and justify your answers.

(a) $\det \begin{bmatrix} q_2 & q_1 & q_3 \end{bmatrix}$
(b) $\det \begin{bmatrix} q_2 & q_3 & q_1 \end{bmatrix}$
(c) $\det \begin{bmatrix} q_2 & 3q_1 & q_3 \end{bmatrix}$
(d) $\det \begin{bmatrix} q_1 + q_2 & q_2 + q_3 & q_3 + q_1 \end{bmatrix}$

(4) A 4×4 matrix A has the same number x in its last row and column, and the other values can be any numbers, i.e.

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} & x \\ a_{21} & a_{22} & a_{23} & x \\ a_{31} & a_{32} & a_{33} & x \\ x & x & x & x \end{bmatrix}$$

(a) The determinant of A is a polynomial in x. What is the degree of the polynomial?
(b) If the top left 3×3 matrix is the identity matrix, which values of x give $\det A = 0$?

(5) Let A be a matrix with eigenvalues $0, 1, 2$ and 3. Suppose the matrix of eigenvalues is the (orthonormal) matrix

$$S = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

(a) What is $\det A$?
(b) What is the easy way to work out S^{-1}?
(c) Write A as a product of three matrices.
(d) Write $(A+I)^{-1}$ as a product of three matrices.

(6) Let A be the matrix

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{bmatrix}$$

(a) Find the eigenvalues and eigenvectors of A.
(b) Explain why $A^{101} = A$. Is $A^{100} = I$?
(c) Find the three diagonal entries of e^{At}.

(7) Suppose the $n \times n$ matrix A has n orthonormal eignevectors $q_1, \ldots q_n$, with positive eigenvalues $\lambda_1, \ldots \lambda_n$.
(a) What are eigenvectors and eigenvalues of A^{-1}? Justify your answer.
(b) Any vector v may be written as a linear combination of the q_i, i.e. $v = c_1q_1 + \cdots + c_nq_n$. What is a quick formula for the c_i using the orthogonality of the q_i?
(c) The solution to $Ax = b$ is also a combination of the eignevectors, i.e.

$$A^{-1}b = d_1q_1 + \cdots + d_nq_n.$$

What is a quick formula for d_1?

(8) Consider the set of vectors $B = \{\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix}\}$.
(a) Is B a basis for $\mathbb{R}^2$? Justify your answer.
(b) Write the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ in terms of the basis B.
(c) Find a matrix for the linear map which changes you from the standard basis to the B basis.

(9) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for $\mathbb{R}^3$, where

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Use the Gram-Schmidt process to find an orthonormal basis.

(10) (a) Explain why every 3×3 matrix has a real eigenvalue.
(b) What does this say about rotations in $\mathbb{R}^3$?
(c) Write down a real 4×4 matrix all of whose eigenvalues are complex.
(d) What does this say about rotations in $\mathbb{R}^4$?

(11) Find a matrix giving projection from $\mathbb{R}^3$ to:

(a) The subspace spanned by $\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \}$. What are the eigenvalues of this matrix?

(b) The subspace spanned by $\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \}$. What are the eigenvalues of this matrix?