**Q1** $A = \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix}$ $\det(A - \lambda I) = \begin{vmatrix} -4-\lambda & 6 \\ -3 & 5-\lambda \end{vmatrix} = (-4-\lambda)(5-\lambda) + 18$

a)

$= \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$ $\lambda = 2, -1.$

b) $\lambda = 2$: $\begin{bmatrix} -6 & 6 \\ -3 & 3 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ +1 \end{bmatrix}.$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "A"] & \mathbb{R}^2 \\ \mathbb{R}^2 \arrow[u, "P"] \arrow[r, "D"'] & \mathbb{R}^2 \arrow[u, "P"'] \end{tikzcd}$$

$\lambda = -1$: $\begin{bmatrix} -3 & 6 \\ -3 & 6 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}$ $v_{-1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$

$D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ $P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$

c) $D^k = \begin{bmatrix} 2^k & 0 \\ 0 & (-1)^k \end{bmatrix}$ $A^k = P D^k P^{-1}$

$= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^k & 0 \\ 0 & (-1)^k \end{bmatrix} \begin{bmatrix} -1 & +2 \\ +1 & -1 \end{bmatrix} = \begin{bmatrix} 2^k & 2(-1)^k \\ 2^k & (-1)^k \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}.$

$= \begin{bmatrix} -2^k + 2(-1)^k & 2^{k+1} - 2(-1)^k \\ -2^k + (-1)^k & 2^{k+1} - (-1)^k \end{bmatrix}.$

d) $2^k, (-1)^k$

**Q2** a) $L = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x - y \\ 2x + y \end{bmatrix}.$

b)

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "L"] & \mathbb{R}^2 \\ \mathbb{R}^2 \arrow[u, "S"] \arrow[r, "M"'] & \mathbb{R}^2 \arrow[u, "S"'] \end{tikzcd}$$

$M = S^{-1} L S^{+1}$

$= \begin{bmatrix} 0 & +1 \\ +1 & +1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -4 & 4 \end{bmatrix}$

c) rank $= 2$, nullity $= 0$

Q3 a) $\det[a_2 a_1 a_3] = -1$ b) $\det[a_2 a_3 a_1] = +1$ ②

c) $\det[a_2\ 3a_1\ a_3] = -3$

d) $\det[a_1+a_2\ \ a_2+a_3\ \ a_3+a_1] = \det[a_1\ \ a_2+a_3\ \ a_3+a_1] + \det[a_2\ \ a_2+a_3\ \ a_3+a_1]$

$= \det[a_1\ a_2\ \ a_3+a_1] + \det[a_1\ a_3\ \ a_3+a_1] + \underbrace{\det[a_2\ a_2,\ a_3+a_1]}_{=0} + \det[a_2\ a_3\ a_3+a_1]$

$= \overset{+1}{\det[a_1\ a_2\ a_3]} + \overset{0}{\cancel{\det[a_1 a_2 a_1]}} + \overset{0}{\cancel{\det[a_1\ a_3\ a_3]}} + \overset{0}{\cancel{\det[a_1\ a_3\ a_1]}}$
$+ \overset{0}{\cancel{\det[a_2\ a_3, a_3]}} + \underset{+1}{\det[a_2\ a_3\ a_1]} = 2.$

Q4 a) $\det(A) = \underset{\sigma \text{ permutation}}{\sum} \prod_{i=1}^{n} a_{i\sigma(i)}. \Rightarrow \det(A)$ deg 2 in $x$.

b)
$$\begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & x \\ x & x & x & x \end{bmatrix} = \begin{vmatrix} 1 & 0 & x \\ 0 & 1 & x \\ x & x & x \end{vmatrix} - x \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x & x & x \end{vmatrix}$$

$= \begin{vmatrix} 1 & x \\ x & x \end{vmatrix} + x \begin{vmatrix} 0 & 1 \\ x & x \end{vmatrix} + x \begin{vmatrix} 0 & 1 \\ x & x \end{vmatrix} = x - x^2 + x(-x) + x(-x)$

$= x - 3x^2 = 0 \quad x(x-3) \Rightarrow x = 0, 3.$

Q5 a) $\det(A) = \pm 1$. $\frac{1}{2}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 2 & 2 \\ 0 & 2 & -2 & 0 \end{bmatrix} \rightsquigarrow \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & -2 & 2 \end{bmatrix}$

$\rightsquigarrow \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$. so $\det(A) = \left(\frac{1}{2}\right)^4 1.2.2.4 = +1$.

b) $\det(A) = 0.1.2.3 = 0$.

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "A"] & \mathbb{R}^2 \\ \mathbb{R}^2 \arrow[u, "S"] \arrow[r, "D"] & \mathbb{R}^2 \arrow[u, "S"'] \end{tikzcd}$$

b) $S^{-1} = S^T$

c) $A = SDS^{-1}$ $\quad D = \begin{bmatrix} 0&0&0&0 \\ 0&1&0&0 \\ 0&0&2&0 \\ 0&0&0&3 \end{bmatrix}$

d) $A+I = SDS^{-1}+I = S(D+I)S^{-1}$

So $(A+I)^{-1} = S(D+I)^{-1}S^{-1} = S\begin{bmatrix} 1&&&0 \\ &2&& \\ &&3& \\ 0&&&4 \end{bmatrix}S^{-1}$.

Q6 a) $A = \begin{bmatrix} 1&3&1 \\ 0&0&-2 \\ 0&0&-1 \end{bmatrix}$ eigenvalues $\lambda = 1, 0, -1$.

b)

$A = SDS^{-1}$ for some $S$. $\quad A^k = SD^kS^{-1}$ $\quad \cancel{A^2}$ $\quad D = \begin{bmatrix} 1&0&0 \\ 0&0&0 \\ 0&0&-1 \end{bmatrix}$.

so $D^2 = \begin{bmatrix} 1&0&0 \\ 0&0&0 \\ 0&0&1 \end{bmatrix}$. so $D^{odd} = \begin{bmatrix} 1&0&0 \\ 0&0&0 \\ 0&0&-1 \end{bmatrix}$ $D^{even} = \begin{bmatrix} 1&0&0 \\ 0&0&0 \\ 0&0&1 \end{bmatrix}$.

$A^{101} = SD^{101}S^{-1} = SDS^{-1} = A$.

$A^{100} \neq I$ as eigenvalues of $I$ are $1,1,1$, eigenvalues of $A^{100}$ $1,1,0$.

c) $e^{At} = I + At + A^2\frac{t^2}{2!} + \cdots \;(*) = S\left(I + Dt + D^2\frac{t^2}{2!} + D^3\frac{t^3}{3!} + \cdots\right)S^{-1}$

$= S\begin{bmatrix} 1+t+\frac{t^2}{2!}+\cdots & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\cdots \end{bmatrix}S^{-1} = S\begin{bmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}S^{-1}$

can find $S$, or easier just compute $(*)$ directly. $\quad A^2 = \begin{bmatrix} 1&3&-6 \\ 0&0&2 \\ 0&0&1 \end{bmatrix}$

$$\begin{bmatrix} \underbrace{1+t+\frac{t^2}{2!}+\cdots}_{e^t} & 3\left(1+t+\frac{t^2}{2!}+\cdots\right) & 1-6t+\frac{t^2}{2!}+\cdots \\ 0 & 0 & -2+2t-\frac{2t^2}{2!}+\cdots \\ 0 & 0 & \underbrace{-1+t-\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots}_{-e^{-t}} \end{bmatrix}.$$

Q7 a) eigenvalues $\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_n^{-1}$

eigenvectors. $q_1\ q_2 \ldots q_n$

check : $\cancel{A^{-1}q_i}$ $\quad A q_i = \lambda_i q_i \Rightarrow \frac{1}{\lambda_i} q_i = A^{-1} q_i$ ✓.

b) $v = c_1 q_1 + c_2 q_2 + \cdots + c_n q_n \qquad c_k = (v \cdot q_k)$.

c) $b = b_1 q_1 + \cdots + b_n q_n$ where $b_k = b \cdot q_k$.

$A^{-1}(b) = A^{-1}(b_1 q_1 + \cdots + b_n q_n) = \frac{b_1}{\lambda_1} q_1 + \cdots + \frac{b_n}{\lambda_n} q_n$.

Q8

a) $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & 5 \\ 0 & 1/3 \end{bmatrix}$ 2 pivots $\Rightarrow \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$ basis for $\mathbb{R}^2$.

b) $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}_E = \begin{bmatrix} 7 \\ -4 \end{bmatrix}_B$.

c) $\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$.

Q9 $v_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \qquad q_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

$q_2' = v_2 - (v_2 \cdot q_1) q_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{6}}\ 1\ \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/6 \\ -7/6 \\ 4/6 \end{bmatrix}$.

$\cancel{q_2 = \frac{\sqrt{66}}{6} \begin{bmatrix} 1/6 \\ -7/6 \\ 4/6 \end{bmatrix}} = \qquad q_2 = \frac{1}{6\sqrt{66}} \begin{bmatrix} 1 \\ -7 \\ 4 \end{bmatrix}$.

$q_3' = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - (v_3 \cdot q_1) q_1 - (v_3 \cdot q_2) q_2$ etc.

Q10 a) $A$ 3x3 $\Rightarrow \det(A-\lambda I)$ cubic $\Rightarrow$ real ~~eig~~ solution

b) every rotation has a fixed direction / axis.

c) $$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix}$$

d) rotation in $\mathbb{R}^4$ don't need to have a fixed direction

Q11 a) $$P = A(A^TA)A^{-1} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \Big( \underbrace{\begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}_{3} \Big)^{-1} \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}.$$

$$= \frac{1}{3} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$ eigenvalues $1, 0, 0$.

b) $$\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \left( \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

$$\left( \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \right)^{-1}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{8} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

$$\frac{1}{8} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & -2 & 2 \\ 4 & 2 & -2 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 8 & 0 & 0 \\ 0 & 4 & -4 \\ 0 & -4 & 4 \end{bmatrix}$$

eigenvalues $1, 1, 0$.