Math 338 Linear Algebra Spring 22 Midterm 2a

Name: Solutions

- I will count your best 8 of the following 10 questions.
- You may use a calculator without symbolic algebra capabilities, and a 3×5 index card of notes.

1	20	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 2	
Overall	

(1) Consider the matrix A with LU factorization: $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{array}{c} \begin{matrix} & s & & t \\ & \downarrow & & \downarrow \end{matrix} \\ \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$

(a) The matrix A determines a map $x \mapsto Ax$ from $\mathbb{R}^a$ to $\mathbb{R}^b$. What are a and b?

(b) Find all solutions to the equation $Ax = 0$.

a) $a = 4$ $\quad b = 3$.

b) $x_4 = t$

$x_3 + x_4 = 0 \qquad x_3 = -t$

$x_2 = s$

$x_1 - x_2 + x_4 = 0 \qquad x_1 = s - t$

$$\left\{ \begin{bmatrix} s-t \\ s \\ -t \\ t \end{bmatrix} \right\} = \left\{ s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

(2) Consider the equation $Ax = b$, where $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -1 & 3 \\ 3 & -2 & 4 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 0 \\ c \end{bmatrix}$. Find all possible solutions for all values of c.

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 1 & -1 & 3 & 0 \\ 3 & -2 & 4 & c \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -\cancel{4}2 & c-3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -\cancel{4}2 & c-3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

unique solution for all values of c

$x_3 = -1$

$x_2 - 2x_3 = c - 3.$ $\qquad x_2 = c - 5$

$x_1 - x_2 + 2x_3 = 1$ $\qquad x_1 - c + 5 - 2 = 1$ $\qquad x_1 = -2 + c$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2+c \\ -5+c \\ -1 \end{bmatrix}.$$

(3) Let S be the following set of vectors.

$$S = \{ \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \}.$$

(a) Find a subset of S which is a basis for the span of S.
(b) Do the vectors in S span $\mathbb{R}^3$?

$$\begin{bmatrix} 3 & 1 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 3 & -1 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 2 & 1 \\ 1 & 2 & 1 & 0 \\ -1 & 3 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & 6 & 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

a) $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$

b) yes.

(4) (a) Write down a basis for $\mathbb{R}^4$.
(b) Write down a spanning set for $\mathbb{R}^4$ which is not a basis.
(c) Write down a linearly independent set in $\mathbb{R}^4$ which is not a basis.

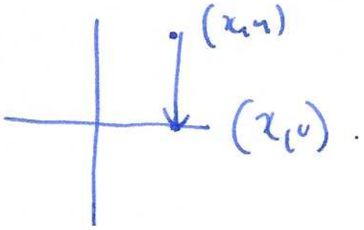
a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$

b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$

c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$

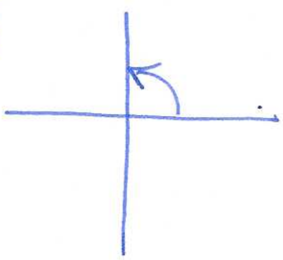
(5) (a) Write down a 2×2 matrix giving projection onto the x-axis.
(b) Write down a 2×2 matrix corresponding to a rotation by $\pi/2$.
(c) Use your answers to (a) and (b) to write down a product of matrices which gives projection onto the y-axis.

a)



$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}.$$

b)



$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \theta = \frac{\pi}{2} \quad R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

c)

$$R A R^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

(6) Use the Gram-Schmidt algorithm to find an orthonormal basis for $\mathbb{R}^3$, starting with the basis $\{\underset{v_1}{\begin{bmatrix}0\\0\\3\end{bmatrix}}, \underset{v_2}{\begin{bmatrix}1\\1\\1\end{bmatrix}}, \underset{v_3}{\begin{bmatrix}1\\2\\3\end{bmatrix}}\}$.

$$q_1 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$q_2' = v_2 - (q_1 \cdot v_2) q_1 = \begin{bmatrix}1\\1\\1\end{bmatrix} - (1)\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$q_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}.$$

$$q_3' = v_3 - (q_1 \cdot v_3) q_1 - (q_2 \cdot v_3) q_2$$

$$= \begin{bmatrix}1\\2\\3\end{bmatrix} - 3\begin{bmatrix}0\\0\\1\end{bmatrix} - \frac{1}{\sqrt{2}}\, 3 \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\2\\0\end{bmatrix} - \frac{3}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}-1/2\\1/2\\0\end{bmatrix}$$

$$q_3 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}.$$

(7) Let $v = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ be a vector in $\mathbb{R}^3$.

(a) Show that the set of all vectors perpendicular to v is a vector subspace of $\mathbb{R}^3$.

(b) Find a basis for this vector subspace.

a) addition : if $x.v = 0$ and $y.v = 0$ then $(x+y).v = x.y + y.v = 0$ ✓

scalar multiplication : if $x.v = 0$ then $(kx).v = k(x.v) = k.0 = 0$ ✓.

b) $\begin{array}{ccc} \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \\ & \uparrow & \uparrow \\ & s & t \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ $\quad x_1 + s - t = 0 \quad x_1 = -s+t$

$x_2 = s$

$x_3 = t$

$\left\{ \begin{bmatrix} -s+t \\ s \\ t \end{bmatrix} \right\} = \left\{ s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

basis for null space / kernel is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(8) Let $A = \begin{bmatrix} 4 & -3 \\ 6 & -5 \end{bmatrix}$.

(a) Find the eigenvalues for A.
(b) Find the eigenvectors for A.

a) $\begin{vmatrix} 4-\lambda & -3 \\ 6 & -5-\lambda \end{vmatrix} = -(4-\lambda)(5+\lambda) + 18$

$= \lambda^2 + \lambda - 20 + 18 = \lambda^2 + \lambda - 2$

$= (\lambda+2)(\lambda-1)$

$\lambda = -2, 1$

b) $\lambda = -2$: $\begin{bmatrix} 6 & -3 \\ 6 & -3 \end{bmatrix} \sim \begin{bmatrix} 6 & -3 \\ 0 & 0 \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda = 1$: $\begin{bmatrix} 3 & -3 \\ 6 & -6 \end{bmatrix} \sim \begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "A"] \arrow[d, leftrightarrow, "S"'] & \mathbb{R}^2 \arrow[d, leftrightarrow, "S"] \\ \mathbb{R}^2 \arrow[r, "D"'] & \mathbb{R}^2 \end{tikzcd}.$$

(9) Let $A = \begin{bmatrix} 4 & -3 \\ 6 & -5 \end{bmatrix}$. Use your answer to the previous question to write down a matrix S, and a diagonal matrix D such that $A = S^{-1}DS$. Verify that this is correct.

$$D = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \qquad S^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad S = -\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}.$$

$$S^{-1}DS = \cancel{-\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}}$$

$$\cancel{= \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -2 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & \\ & \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1+1 \\ +2-1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 6 & -5 \end{bmatrix} \checkmark.$$

(10) Let P_3 be the vector space of cubic polynomials.

(a) Write down a basis for P_3.

(b) Show that the map $T\colon P_3 \to P_3$ given by $f(x) \mapsto xf'(x)$ is a linear map.

(c) Write down a matrix for T with respect to the basis in (a).

a) $\cancel{\{t^3, t^2, t, 1\}}$ $\{x^3, x^2, x, 1\}$.

b) $\cancel{at^3 + bt^2 +}$ sums: $f(x) + g(x) \mapsto x(f'(x) + g'(x))$.

$= xf'(x) + xg'(x)$

$= T(f) + T(g)$. ✓.

constant multiples: $T(kf(x)) = x(kf(x))' = xkf'(x) = k(xf'(x)) = kT(x)$ ✓

c) $T(ax^3 + bx^2 + cx + d) = x(a.3x^2 + b.2x + c) = 3ax^3 + 2bx^2 + cx$

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 3a \\ 2b \\ c \\ 0 \end{bmatrix}$$