Linear Algebra Spring 22 Sample Midterm 2

(1) Suppose A is an $m \times n$ matrix for which $Ax = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ has no solutions and $Ax = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ has exactly one solution.

(a) Give all possible information about m and n and the rank r of A.
(b) Find all solutions to Ax = 0 and explain your answer.
(c) Write down an example of a matrix A that fits the description in part (a).

(2) This 3×4 matrix depends on c:

$$A = \begin{bmatrix} 1 & 2 & 4 & 4 \\ 2 & c & 4 & 6 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

(a) For each c find a basis for the column space of A.
(b) For each c find a basis for the nullspace of A.
(c) For each c find the complete solution x to $Ax = \begin{bmatrix} 1 \\ c \\ 0 \end{bmatrix}$.

(3) If A is a 3 by 5 matrix, what information do you have about the nullspace of A?

(4) Suppose row operations on A lead to this matrix $R = \texttt{rref}(A)$:

$$R = \begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Write all known information about the columns of A.

(5) (a) What can you say about the number of vectors in a spanning set for $\mathbb{R}^n$ which is not a basis?
(b) What can you say about the number of vectors in a linearly independent set in $\mathbb{R}^n$ which is not a basis?

(6) Let S be the following set of vectors.

$$S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

(a) Find a subset of S which is a basis for the span of S.
(b) Do the vectors in S span $\mathbb{R}^3$?

(7) Let v be a vector in $\mathbb{R}^n$. Show that the collection of all vectors perpendicular to v forms a vector space.

(8) $A = \begin{bmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 3 & -1 & -3 & 0 \\ 2 & 1 & -1 & 1 & 2 \end{bmatrix}$

(a) Find the row echelon form for A.
(b) Find the rank and nullity of A.
(c) Find a basis for the image of A.
(d) Find a basis for the kernel of A.

(9) (a) Write down a 2×2 matrix giving a rotation about angle $\pi/3$ anti-clockwise.
(b) Write down a 2×2 matrix giving reflection in the x-axis.
(c) Use your previous answers to write down a 2×2 matrix giving reflection in the line through the origin making an angle of $\pi/3$ with the x-axis.

(10) Consider the vector $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ in $\mathbb{R}^2$ with respect to the standard basis. Write it down with respect to the basis $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

(11) Use the Gram-Schmidt algorithm to produce an orthonormal basis for $\mathbb{R}^3$, starting with the basis $\left\{ \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \right\}$

(12) Let A be the matrix $A = \begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix}$.

(a) Find the eigenvalues of A.
(b) Diagonalize A, i.e. find matrices P and D such that $P^{-1}AP = D$.
(c) Find an exact formula for A^k.

(d) What are the eigenvalues for A^k?

(13) Let $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $L(x, y) = (x + 2y, x - 2y)$. Let

$$F = \{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \}$$

be a basis for $\mathbb{R}^2$, and let E be the standard basis for $\mathbb{R}^2$.

(a) Find the matrix for L with respect to E.
(b) Find the matrix for L with respect to F.
(c) What is the rank and nullity of L?

(14) Let M_2 be the vector space of 2×2 matrices, and consider the map $T\colon A \mapsto A - A^T$.

(a) Show that T is a linear map.
(b) Write down a matrix for T with respect to the standard basis for M_2.
(c) Write down a matrix for T with respect to the basis
$\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \}.$
(d) Is T one-to-one? Explain.
(e) Is T onto? Explain.