SMT2 Solutions

Q1a) $Ax = \begin{bmatrix}1\\0\\1\end{bmatrix}$ no solutions $\Rightarrow$ A not onto, so image(A) has dim ≤ 2
so rank(A) ≤ 2

$Ax = \begin{bmatrix}0\\1\\0\end{bmatrix}$ exactly one solution $\Rightarrow$ A one-to-one, so no free variables, so #cols ≤ 2.

$\underset{m\times n}{A}$ then rank(A) $= r \leq 2$ and #cols $= n = r$

$$\underset{3\times r}{A}\ \underset{r\times 1}{x} = \underset{3\times 1}{\begin{bmatrix}\cdot\\ \cdot\\ \cdot\end{bmatrix}}$$ when $r = 1,2$.

Q1 b) A one-to-one $\Rightarrow$ $Ax=0 \Rightarrow x=0$

c) $A = \begin{bmatrix}0\\1\\0\end{bmatrix}$

Q2 a) $A = \begin{bmatrix}1 & 2 & 4 & 4\\ 2 & c & 4 & 6\\ 0 & 0 & 2 & 1\end{bmatrix} \overset{\text{row ops}}{\sim} \begin{bmatrix}1 & 2 & 4 & 4\\ 0 & c-4 & -4 & -2\\ 0 & 0 & 2 & 1\end{bmatrix}$

$c\neq 4$: 3 pivots, basis is $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\c\\0\end{bmatrix}\begin{bmatrix}4\\4\\2\end{bmatrix}\right\}$ $c=4$ 2 pivots basis $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}4\\4\\2\end{bmatrix}\right\}$.

b) $c\neq 4$ $x_4 = t$

$2x_3 + t = 0$ $x_3 = -\frac{1}{2}t$

$(c-4)x_2 - 4(-\frac{1}{2}t) - 2t = 0$ $x_2 = 0$

$x_1 + 4(-\frac{1}{2}t) + 4t = 0$ $x_1 = -2t$

solutions $\left\{t\begin{bmatrix}-2\\0\\-1/2\\1\end{bmatrix}\right\}$.

$c=4$ $x_4 = t$
$x_2 = s$

$-4x_3 - 2t = 0$ $x_3 = -\frac{1}{2}t$

$x_1 + 2s + 4(-\frac{1}{2}t) + 4t = 0$ $x_1 = -2s - 2t$

solutions $\left\{s\begin{bmatrix}-2\\1\\0\\0\end{bmatrix} + t\begin{bmatrix}-2\\0\\-1/2\\1\end{bmatrix}\right\}$

c) $\left[\begin{array}{cccc|c}1 & 2 & 4 & 4 & 1\\ 2 & c & 4 & 6 & c\\ 0 & 0 & 2 & 1 & 0\end{array}\right] \xrightarrow{\text{row ops}} \left[\begin{array}{cccc|c}1 & 2 & 4 & 4 & 1\\ 0 & c-4 & -4 & -2 & c-2\\ 0 & 0 & 2 & 1 & 0\end{array}\right]$.

~~$c\neq 4$ particular solution:~~

$c\neq 4$ find particular solution: choose $x_4 = 0$

$2x_3 = 0$ $(c-4)x_2 = c-2$ $x_2 = \frac{c-2}{c-4}$

$x_1 + \frac{2(c-2)}{c-4} = 1 \quad x_1 = 1 - \frac{2(c-2)}{c-4}$ general sol^n. $\begin{bmatrix} 1-\frac{2(c-2)}{c-4} \\ \frac{c-2}{c-4} \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ -1/2 \\ 1 \end{bmatrix}$

$\underline{c=4}$ $\left[\begin{array}{cccc|c} 1 & 2 & 4 & 4 & 1 \\ 0 & 0 & -4 & -2 & c-2=2 \\ 0 & 0 & 0 & 0 & -1/2 \end{array}\right]$ inconsistent, no solutions

Q3 A $\underset{\text{rows cols}}{3\times 5}$. rank$(A)$ = #pivots ≤ 3

#cols $= 5 \geq 3 \Rightarrow$ # free vars ≥ 2

so $2 \leq \dim(\text{null}(A)) \leq 5$.

Q4 rank$(A) = 3$, columns 1,3,5 form basis for column space, cols 2,4 sum of cols 1,3,5.

Q5 a) # vectors $> n$ b) #vectors $< n$.

Q6 $\begin{bmatrix} 1 & 3 & 2 & 1 \\ -1 & 0 & 1 & 2 \\ 2 & 5 & 3 & 1 \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 3 & 3 & 3 \\ 0 & -1 & -1 & -1 \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

a) basis for span is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} \right\}$. b) no rank(span(A)) $= 2 < 3$.

Q7 let $V = \{x \in \mathbb{R}^n, x.v = 0\}$ check vector space.

if $x, y \in V$ then $(x+y).v = x.v + y.v = 0 + 0 = 0$ ✓.

if $~~k~~$ $x \in V$ $k \in \mathbb{R}$, then $kx.v = k(x.v) = k(0) = 0$. ✓.

Q8

a) $\begin{bmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 3 & -1 & 3 & 0 \\ 2 & 1 & -1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 3 & -1 & 3 & 0 \\ 0 & 3 & -1 & -3 & -2 \end{bmatrix} \sim \begin{array}{c} \begin{bmatrix} 1 & -1 & 0 & 2 & 1 \\ 0 & 3 & -1 & 3 & 0 \\ 0 & 0 & 0 & -6 & -2 \end{bmatrix} \\ \begin{matrix} & & \uparrow s & & \uparrow t \end{matrix} \end{array}$

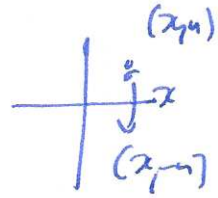
b) rank$(A) = 3$ nullity$(A) = 2$

c) basis for image $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \right\}$

d) $x_5 = t$, $-6x_4 - 2t = 0$ $x_4 = -\frac{1}{3}t$ $x_3 = s$ $3x_2 - s - t = 0$ $x_2 = \frac{1}{3}s + \frac{1}{2}t$

$x_1 - \frac{1}{3}s - \frac{1}{3}t + -\frac{1}{3}t + t = 0 \qquad x_1 = \frac{1}{3}s - \frac{1}{3}t$

solutions: $\left\{ s\begin{bmatrix} 1/3 \\ 1/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1/3 \\ 1/3 \\ 0 \\ -1/3 \\ 1 \end{bmatrix} \right\}$ so basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 3 \end{bmatrix} \right\}$.



Q9

a) $\underbrace{\begin{bmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix}}_{R} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$ b) $\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{T}$

c) $RTR^{-1} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$

$= \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$.

Q10 $x_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ $\qquad \left[\begin{array}{cc|c} 2 & 1 & 2 \\ 1 & 1 & -3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & -3 \\ 0 & -1 & 8 \end{array}\right]$

$x_2 = -8 \qquad x_1 - 8 = -3 \qquad x_1 = 5 \qquad \begin{bmatrix} 5 \\ -8 \end{bmatrix}_{\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}}$.

Q11 $\left\{ \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \right\}$. $\qquad q_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

$q_2' = v_2 - (v_2 \cdot q_1)q_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad q_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

$q_3' = v_3 - (v_3 \cdot q_1)q_1 - (v_3 \cdot q_2)q_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$q_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

Q12 a) $A = \begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix} \qquad \det(A - \lambda I) = \begin{vmatrix} (5-\lambda) & -3 \\ 6 & (-4-\lambda) \end{vmatrix} = (\lambda-5)(\lambda+4) + 18$

$= \lambda^2 - \lambda - 2$

$= (\lambda - 2)(\lambda + 1) \qquad \lambda = 2, -1$.

b) $\lambda = 2$: $(A-2I)x = 0$ $\quad \begin{bmatrix} 3 & -3 \\ 6 & -6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

$\lambda = -1$: $(A+I)x = 0$ $\quad \begin{bmatrix} 6 & -3 \\ 6 & -3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "A"] & \mathbb{R}^2 \\ \mathbb{R}^2 \arrow[u] \arrow[r, "D"] & \mathbb{R}^2 \arrow[u, "S"'] \end{tikzcd} \quad S = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

$D = S^{-1}AS \qquad D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

$S = P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$

$A = SDS^{-1} \qquad A^k = SD^kS^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2^k & 0 \\ 0 & (-1)^k \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$

$= \begin{bmatrix} 2^k & (-1)^k \\ 2^k & 2(-1)^k \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 2^k - (-1)^k & -2^k + (-1)^k \\ 2 \cdot 2^k - 2(-1)^k & -2^k + 2(-1)^k \end{bmatrix}.$

eigenvalues for A^k at $2^k, (-1)^k$.

Q13 a) $L(x,y) = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

b)
$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "L"] & \mathbb{R}^2 \\ \mathbb{R}^2 \arrow[u, "S"] \arrow[r, "A"'] & \mathbb{R}^2 \arrow[u, "S"'] \end{tikzcd}$$

$L = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix} \qquad S = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}.$

so $A = S^{-1}LS = -\begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix}.$

c) $\begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix} \overset{\text{row ops}}{\rightsquigarrow} \begin{bmatrix} 1 & 2 \\ 0 & -4 \end{bmatrix}$ 2 pivots so rank $= 2$
nullity $= 0$.

Q14 $\quad M_2 \quad T: M_2 \longrightarrow M_2 \quad A \mapsto A - A^T$

a) $T(A+B) = (A+B) - (A+B)^T = A - A^T + B - B^T = T(A) + T(B)$ ✓

$T(kA) = kA - (kA)^T = k(A - A^T) = kT(A)$ ✓.

b) standard basis $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & b-c \\ c-b & 0 \end{bmatrix}$$

matrix is
$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ b-c \\ c-b \\ 0 \end{bmatrix}$$

c)
$$\begin{tikzcd} \mathbb{R}^4 \arrow[r, "T"] & \mathbb{R}^4 \\ \mathbb{R}^4 \arrow[u, "S"] \arrow[r, "A"'] & \mathbb{R}^4 \arrow[u, "S"'] \end{tikzcd}$$

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad S^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$A = S^{-1} T S$ ← do matrix mult

alternate method: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ so matrix is

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}.$$