MTH 338 Linear Algebra Spring 22 Midterm 1a

Name: Solutions

- You must do Q1. Do any 7 of the following 9 questions.
- You may use a calculator without symbolic algebra capabilities, and a 3×5 index card of notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 1	
Overall	

(1) For which values of k do the following equations have more than one solution? Find all solutions for that value of k.

$$\begin{aligned} x_1 - x_2 + \ \ x_3 &= -1 \\ x_1 + x_2 + 2x_3 &= \ \ 0 \\ 2x_1 \qquad + kx_3 &= -1 \end{aligned}$$

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 2 & 0 \\ 2 & 0 & k & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 2 & 1 & 1 \\ 0 & 2 & k-2 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{1} & -1 & 1 & -1 \\ 0 & \boxed{2} & 1 & 1 \\ 0 & 0 & k-3 & 0 \end{bmatrix}$$

$k \neq 3$: 3 pivots, unique solution

$k = 3$: 2 pivots, infinitely many solutions

$x_3 = t$

$2x_2 + x_3 = 1 \qquad x_2 = \frac{1}{2} - \frac{1}{2}t$

$x_1 - x_2 + x_3 = -1 \qquad x_1 = -1 - t + \frac{1}{2} - \frac{1}{2}t = -\frac{1}{2} - \frac{3}{2}t$

$$\left\{ \begin{bmatrix} -\frac{1}{2} \\ 1/2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3/2 \\ -1/2 \\ 1 \end{bmatrix} \right\}$$

(2) Find all solutions to $Ax = 0$, where A is the matrix whose LU factorization is given below.

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ -2 & -4 & 4 & -5 & -3 \\ 1 & 2 & 3 & 3 & 3 \\ -1 & -2 & 5 & 5 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ -1 & 2 & 3 & 1 \end{bmatrix} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} \boxed{1} & 2 & -1 & 2 & 1 \\ 0 & 0 & \boxed{2} & -1 & -1 \\ 0 & 0 & 0 & \boxed{3} & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} & \uparrow & & & \uparrow \\ & \text{free var } s & & & \text{free var } t \end{array} \end{array}$$

$x_5 = t$

$3x_4 + 4x_5 = 0 \qquad x_4 = -\frac{4}{3}t$

$2x_3 - x_4 - x_5 = 0 \qquad 2x_3 = \frac{4}{3}t + t = \frac{7}{3}t \qquad x_3 = \frac{7}{6}t$

$x_2 = s$

$x_1 + 2x_2 - x_3 + 2x_4 + x_5 = 0 \qquad x_1 = -2s + \frac{7}{6}t + \frac{\overset{16}{8}}{\underset{6}{3}}t - t = -2s + \frac{17}{6}t$

$$\left\{ s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 17/6 \\ 0 \\ 7/6 \\ -4/3 \\ 1 \end{bmatrix} \right\}.$$

(3) Write down a basis for the column space of the matrix A, whose LU factorization is given below.

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ -2 & -4 & 4 & -5 & -3 \\ 1 & 2 & 3 & 3 & 3 \\ -1 & -2 & 5 & 5 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ -1 & 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} \boxed{1} & 2 & -1 & 2 & 1 \\ 0 & 0 & \boxed{2} & -1 & -1 \\ 0 & 0 & 0 & \boxed{3} & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

basis for column space $\leftrightarrow$ pivot cols of U

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 3 \\ 5 \end{bmatrix} \right\}$$

(4) Write down a basis for the row space of the matrix A, whose LU factorization is given below.

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ -2 & -4 & 4 & -5 & -3 \\ 1 & 2 & 3 & 3 & 3 \\ -1 & -2 & 5 & 5 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ -1 & 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\{[1\ 2\ {-1}\ 2\ 1], [0\ 0\ 2\ {-1}\ {-1}], [0\ 0\ 0\ 3\ 4]\}$

(5) Are the following vectors linearly independent?

$$\{[1,-2,1,-1],[-1,4,3,5],[2,-5,3,5],[1,-3,3,9]\}$$

Hint: you may use the LU factorization from the previous problems, as long as you explain your reasoning clearly.

No - first three vectors are a basis for the column space, so the fourth is a linear combination of the first three, so they are not independent.

(6) Suppose A and B are invertible matrices.

(a) Is $A^{-1}BA$ invertible? If so, write down the inverse.

(b) Is $B - A$ invertible? Explain why or give a counterexample.

a) $(A^{-1}BA)^{-1} = A^{-1}B^{-1}A$

b) no, choose $A = B = I$ identity invertible,

then $B - A = I - I = 0$ not invertible.

(7) Give an example of a system of three equations in three unknowns which is inconsistent.

$$x_1 = 0$$

$$x_2 = 0$$

$$0 \cdot x_3 = 1$$

(8) (a) If a system has two equations and three unknowns can it be inconsistent? Give an example or justify your answer.

(b) If a system has two equations and three unknowns can there be a unique solution? Justify your answer.

a) yes :

$$x_1 + x_2 + x_3 = 0$$
$$x_1 + x_2 + x_3 = 1$$

b) no : the system will either be inconsistent, or has at most two pivots, so there will be at least one free variable, so there will be infinitely many solutions.

(9) Let P_2 be the vector space of quadratic polynomials, and consider the map $f\colon P_2 \to \mathbb{R}$ given by $p(x) \mapsto p(4)$. Show that f is a linear map, and find the kernel, i.e. describe the subspace of vectors with $f(p) = 0$.

P_2 $\{ax^2+bx+c\}$ basis $\{x^2, x, 1\}$

$f: ax^2+bx+c \longmapsto 16a+4b+c$

check linear : $f(p+q) = (p+q)(4) = p(4)+q(4) = f(p)+f(q)$

$f(kp) = kp(4) = kf(p)$

find kernel :

$$\begin{array}{c} \begin{bmatrix} \boxed{16} & 4 & 1 \end{bmatrix} \\ \text{pivot} \quad \uparrow \; \uparrow \\ \quad\; s \;\; t \\ \text{free vars.} \end{array} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 16a+4b+c \end{bmatrix}$$

$c = t$

$4b = s$

$16a+4b+c = 16a+s+t = 0 \qquad a = -\frac{1}{16}s - \frac{1}{16}t$

$$\left\{ s \begin{bmatrix} -1/16 \\ 1/4 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1/16 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(10) Let M_2 be the vector space of 2×2 matrices, and consider the map $S\colon M_2 \to M_2$ given by $S(A) = A + A^T$. Show that S is a linear map, and write out a matrix for S with respect to a clearly stated basis.

check linear: $S(A+B) = (A+B) + (A+B)^T = A + B + A^T + B^T$

$= (A+A)^T + B + B^T = S(A) + S(B)$

$S(kA) = (kA)^T = kA^T = k\,S(A)$.

basis for M_2: $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$$S\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2a \\ b+c \\ b+c \\ 2d \end{bmatrix}$$