Linear Algebra Spring 22 Sample midterm 1

(1) Explain why the following statements are true, or give a counterexample.
  (a) If A and B are invertible $n \times n$ matrices, then AB is also invertible.
  (b) If A and B are invertible $n \times n$ matrices, then $A + B$ is also invertible.
  (c) An invertable matrix has a unique inverse.
  (d) If A and B are symmetric matrices, then AB is also symmetric.
  (e) If A and B are symmetric matrices, then $A + B$ is also symmetric.

(2) Write "impossible" or give an example of:
  (a) A 3×3 matrix with no zeros but which is not invertible.
  (b) A system with two equations and three unknowns that is inconsistent.
  (c) A system with two equations and three unknowns that has a unique solution.
  (d) A system with two equations and three unknowns that has infinitely many solutions.

(3) Consider the following linear system:

$$\begin{cases} 3x_1 + x_2 + 2x_3 & = 2 \\ 3x_2 + 2x_2 + 3x_3 & = 3 \\ 6x_1 \qquad\quad + 2x_3 & = k \end{cases}$$

  (a) For what value of k are there any solutions? Find the solutions explicitly for this value of k.
  (b) If this is written in the form $Ax = b$, write out bases for the four subspaces association to A, for the value of k found above for which there are solutions.

(4) Consider the following linear system:

$$\begin{cases} 3x_1 + 2x_2 + x_3 & = -3 \\ 3x_2 + 4x_2 + 2x_3 & = 0 \\ 6x_1 + 6x_2 + 4x_4 & = -2 \end{cases}$$

Use row reduction to solve these equations.

(5) Let

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 3 & 4 & 2 \\ 6 & 6 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Compute $A - B$, BA, B^T , $\det(A)$. Recall that the determinant is the product of the pivots in the row-echelon form, and use your answer from the previous question.

(6) Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

Find the LU factorization of A.

(7) Use row operations to find the inverse of:

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -2 \\ -2 & 1 & 3 \end{bmatrix}$$

(8) Find examples of 2×2 matrices such that
  (a) $A^2 = -I$
  (b) $A^2 = 0$, with $A \neq 0$
  (c) $CD = -DC \neq 0$
  (d) $EF = 0$, though no entries of E or F are zero.

(9) Are the following vectors linearly independent?
  (a) { [3, 2, 1], [0,0,0], [2, -1, 0] }
  (b) { [1, 2, 3], [1, 0, -1], [2, 2, 2] }
  Are any of the sets bases for $\mathbb{R}^3$? Explain why or why not.

(10) Let P_3 be the vector space of cubic polynomials, let P_2 be the vector space of quadratic polynomials, and let P_1 be the vector space of linear polynomials.
  (a) Write down a basis for P_3.
  (b) Write down a basis for P_2.
  (c) Show that the map $D_1 \colon P_3 \to P_2$ which sends a polynomial to its derivative is a linear map, and so given by $x \mapsto Ax$ for some matrix A. Find the matrix A explicitly.
  (d) Find the image of P_3 in P_2 under the map D_1.
  (e) Find the kernel of D_1.
  (f) Consider the map $D_2 \colon P_3 \to P_1$ which send a polynomial to its second derivative. Write down a basis for P_1 and write down the matrix for D_2 in terms of this basis and the previous basis for P_3.
  (g) Find the kernel of D_2.

(11) Write out explicit bases for the subspace of 2×2 matrices consisting of upper triangular matrices, and the subspace of lower triangular matrices. Explain why they can't be orthogonal subspaces for any choice of inner product.