MTH 338 SMT 1 Solutions

Q1 a) true $(AB)^{-1} = B^{-1}A^{-1}$, check $ABB^{-1}A^{-1} = AA^{-1} = I$.

b) false $\pm I$, invertible, but $I - I = 0$ not invertible.

c) true, if A, B are inverses for C then $\underbrace{(AC)}_{I}B = A\underbrace{(CB)}_{I} \Rightarrow B = A$.

d) false $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

e) true $(A+B)^T = A^T + B^T = A + B$.

Q2 a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ b) $\begin{matrix} x+y+z = 0 \\ x+y+z = 1 \end{matrix}$ c) impossible d) $\begin{matrix} x+y+z=0 \\ y+z=0 \end{matrix}$

Q3 a) $\left[\begin{array}{ccc|c} 3 & 1 & 2 & 2 \\ 3 & 2 & 3 & 3 \\ 6 & 0 & 2 & k \end{array}\right]$ $\left[\begin{array}{ccc|c} 3 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -2 & -2 & k-4 \end{array}\right]$ $\left[\begin{array}{ccc|c} 3 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & k-2 \end{array}\right]$

no solutions unless $k=2$. If $k=2$, $x_3 = t$, $x_2 + t = 1$ $x_2 = 1-t$

$3x_1 + (1-t) + 2t = 2$, $x_1 = \frac{1}{3} - \frac{t}{3}$ so $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1-t)/3 \\ 1-t \\ t \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1/3 \\ -1 \\ 1 \end{bmatrix}$.

b) $N(A)$ $\left\{ \begin{bmatrix} -1/3 \\ -1 \\ 1 \end{bmatrix} \right\}$. row$(A)$ $\{ [3,1,2], [0,1,1] \}$.

$\text{col}(A) = \left\{ \begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$. $N(A^T)$ $A^T = \begin{bmatrix} 3 & 3 & 6 \\ 1 & 2 & 0 \\ 2 & 3 & 2 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 6 \\ 0 & -1 & 2 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$x_3 = t$, $x_2 + 2t = 0$, $x_2 = -2t$

$x_1 + 2(-2t) = 0$ $x_1 = 4t$ $\left\{ \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \right\}$.

Q4 $\left[\begin{array}{ccc|c} 3 & 2 & 1 & -3 \\ 3 & 4 & 2 & 0 \\ 6 & 6 & 4 & 2 \end{array}\right]$ $\left[\begin{array}{ccc|c} 3 & 2 & 1 & -3 \\ 0 & 2 & 1 & 3 \\ 0 & 2 & 2 & 4 \end{array}\right]$ $\left[\begin{array}{ccc|c} 3 & 2 & 1 & -3 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$.

$x_3 = 1$ $2x_2 + \underbrace{x_3}_{1} = 3$ $x_2 = 1$ $3x_1 + 2 + 1 = -3$ $x_1 = -2$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

Q5 $A = \begin{bmatrix} 3 & 2 & 1 \\ 3 & 4 & 2 \\ 6 & 6 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ $A - B = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 5 & 1 \\ 6 & 5 & 5 \end{bmatrix}$

$BA = \begin{bmatrix} 3 & -4 & -4 \\ 6 & 6 & 3 \\ -3 & -2 & -2 \end{bmatrix}$ $B^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$ $\det(A) = 3 \times 2 \times 1 = 6.$

Q6 $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}.$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

Q7
$$\left[\begin{array}{ccc|ccc} 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 \\ -2 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 2 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right].$$
$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & \cancel{2} & -2 & 1 & 0 \\ 0 & 1 & 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right]$$

Q8 a) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ b) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ c) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$ d) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Q9 a) no b) $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 2 \\ 3 & -1 & 2 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -2 \\ 0 & -4 & -4 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ dependent.

~~need~~ no, as not independent, so not bases.

Q10 a) $\{t^3, t^2, t, 1\}$ b) $\{t^2, t, 1\}$ c) $at^3 + bt^2 + ct + d \xmapsto{D_1} 3at^2 + 2bt + c$

so $D_1 = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 3a \\ 2b \\ c \end{bmatrix}$ d) image is all of P_2.

e) kernel has basis $\{1\}$ f) P_1 $\{t, 1\}$ basis $D_2 : at^3 + bt^2 + ct + d \xmapsto{D_2} 6at + 2b$

$$\begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 6a \\ 2b \end{bmatrix}$$

g) kernel: basis $\{1, t\}$.

<u>Q11</u> $U \subseteq M_2$ has basis $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$L \subseteq M_2$ has basis $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

can't be orthogonal as $U \cap L \neq \{\underline{0}\}$.