Linear Algebra Spring 22 Sample Final

(1) Find all solutions to the following system of linear equations.

$$\begin{aligned} 2x_1 - 2x_2 + 2x_3 + 2x_4 &= 0 \\ 2x_1 + x_3 + 3x_4 &= 0 \\ x_1 - x2 + x_3 + 2x_4 &= 0 \end{aligned}$$

(2) (a) Write down a matrix for a linear transformation of $\mathbb{R}^3$ which rotates by $\pi/2$ clockwise about the z-axis.
(b) Write down a matrix for a linear transformation of $\mathbb{R}^3$ which doubles lengths in the y-direction.
(c) Write down a matrix for a linear transformation of $\mathbb{R}^3$ which reflects in the yz-plane.
(d) Use your answers above to write down a matrix for a linear transformation of $\mathbb{R}^3$ which doubles lengths in the y-direction, then rotates by $\pi/2$ clockwise about the z-axis, and then reflects in the yz-plane.

(3) (a) Write down a spanning set for $\mathbb{R}^4$ which is not linearly independent.
(b) Write down a basis for $\mathbb{R}^4$ which is not orthogonal.
(c) Write down a set of three vectors which span a two-dimensional subspace of $\mathbb{R}^4$.

(4) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for $\mathbb{R}^3$, where

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Use the Gram-Schmidt process to find an orthogonal basis.

(5) Let $V = \text{Span}\{(1, -2, 1), (3, 1, 2), (2, 3, 1)\}$ in $\mathbb{R}^3$.
(a) What is the dimension of V?
(b) Find a basis for $V^\perp$.

(6)

$$A = \begin{bmatrix} 12 & 15 \\ -10 & -13 \end{bmatrix}$$

(a) Find the eigenvalues of A.
(b) Find the eigenvectors for A.
(c) Diagonalize A, i.e. find matrices P and D such that $P^{-1}AP = D$.
(d) Write down products of matrices which give you A^k and e^{At}.

(7) Let $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $L(x,y) = (3x - y, x + 2y)$. Let

$$S = \left\{ \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

be a basis for $\mathbb{R}^2$, and let T be the standard basis for $\mathbb{R}^2$.

(a) Find the matrix for L with respect to T.
(b) Find the matrix for the change of basis from S to T.
(c) Find the matrix for L with respect to S. Don't worry if its not diagonal.

(8) If A is a non-singular $n \times n$ matrix such that $A^{-1} = A^T$, what can you say about the determinant of A?

(9) Let A be a 4×5 matrix such that the sum of the rows add up to the zero vector.

(a) What can you say about the column rank of A?
(b) The matrix A determines a linear map $L : \mathbb{R}^5 \to \mathbb{R}^4$ given by $L(\mathbf{x}) = A\mathbf{x}$, what can you say about the kernel of L?

(10) Let A have the following LU factorization:

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ -2 & 2 & -3 & -1 \\ 1 & -1 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Find all solutions to $Ax = 0$.
(b) Find all solutions to $Ax = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.
(c) Find a basis for the columns space.
(d) Find a basis for the row space.

(11) Let V be the vector space of 2×2 matrices under addition and scalar multiplication.

(a) Write down a basis for V.
(b) Let $\mathrm{tr}(A)$ be the sum of the diagonal elements of A. Show that the map $A \mapsto \mathrm{tr}(A)$ is a linear map, and find the kernel of this map.
(c) Show that the map $A \mapsto A - \frac{1}{2}\mathrm{tr}(A)I$ is a linear map. What is the image of this map? What is the kernel of this map?