Sample final solutions

Q1

$$\begin{bmatrix} 2 & -2 & 2 & 2 \\ 2 & 0 & 1 & 3 \\ 1 & -1 & 1 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$x_4 = 0$

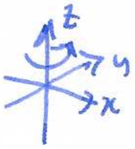
$x_3 = t$

$2x_2 - t + 0 = 0 \qquad x_2 = \frac{1}{2}t$

$x_1 - \frac{1}{2}t + t + 0 = 0 \qquad x_1 = -\frac{1}{2}t$

solutions: $\left\{ t \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} \right\}$

Q2 a)



$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \theta = -\frac{\pi}{2}: \quad \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ c) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \phi \end{bmatrix}$

d)

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \phi \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{2} & -2/\sqrt{2} & 0 \\ -1/\sqrt{2} & 2/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Q3 a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

Q4 $q_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}-1\\1\\1\end{bmatrix}$

$$q_2' = \begin{bmatrix}2\\1\\-1\end{bmatrix} - (q_1 \cdot v_2)\,q_1 = \begin{bmatrix}2\\1\\-1\end{bmatrix} - \frac{1}{\sqrt{3}}(-2)\frac{1}{\sqrt{3}}\begin{bmatrix}-1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\1\\-1\end{bmatrix} + \frac{2}{3}\begin{bmatrix}-1\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}4/3\\5/3\\-1/3\end{bmatrix} \sim \begin{bmatrix}4\\5\\-1\end{bmatrix} \qquad q_2 = \frac{1}{\sqrt{42}}\begin{bmatrix}4\\5\\-1\end{bmatrix}$$

$$q_3' = \begin{bmatrix}1\\1\\1\end{bmatrix} - (q_1 \cdot v_3)\,q_1 - (q_2 \cdot v_3)\,q_2 = \begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{1}{\sqrt{3}}(1)\frac{1}{\sqrt{3}}\begin{bmatrix}-1\\1\\1\end{bmatrix} - \frac{8}{42}\begin{bmatrix}4\\5\\-1\end{bmatrix}$$

$$= \begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{1}{3}\begin{bmatrix}-1\\1\\1\end{bmatrix} - \frac{4}{21}\begin{bmatrix}4\\5\\-1\end{bmatrix} = \frac{1}{21}\begin{bmatrix}21 & +7 & -16\\21 & -7 & -20\\21 & -7 & +5\end{bmatrix} = \frac{1}{21}\begin{bmatrix}12\\-6\\18\end{bmatrix}$$

$$q_3' \sim \begin{bmatrix}2\\-1\\3\end{bmatrix} \qquad q_3 = \frac{1}{\sqrt{14}}\begin{bmatrix}2\\-1\\3\end{bmatrix}$$

Q5

a) $\begin{bmatrix}1 & 3 & 2\\-2 & 1 & 3\\1 & 2 & 1\end{bmatrix} \to \begin{bmatrix}1 & 3 & 2\\0 & 7 & 7\\0 & -1 & -1\end{bmatrix} \sim \begin{bmatrix}1 & 3 & 2\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}$ $\dim V = 2$

b) solve $\begin{bmatrix}1 & -2 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0$ etc : $\begin{bmatrix}1 & -2 & 1\\3 & 1 & 2\\2 & 3 & 1\end{bmatrix} \to \begin{bmatrix}1 & -2 & 1\\0 & 7 & -1\\0 & 7 & -1\end{bmatrix}$

$\sim \begin{bmatrix}1 & -2 & 1\\0 & 7 & -1\\0 & 0 & 0\end{bmatrix}$

$x_3 = t$

$7x_2 - t = 0$

$x_2 = \frac{1}{7}t$

$x_1 - \frac{2}{7}t + t = 0$

$x_1 = -\frac{5}{7}t$

$\begin{bmatrix}-5/7\\1/7\\7\end{bmatrix}t$

can choose basis: $\left\{\begin{bmatrix}-5\\1\\7\end{bmatrix}\right\}$

Q6 $A = \begin{bmatrix} 12 & 15 \\ -10 & -13 \end{bmatrix}$ ③

a) $\det(A-\lambda I) = \begin{vmatrix} 12-\lambda & 15 \\ -10 & -13-\lambda \end{vmatrix} = (12-\lambda)(-13-\lambda) + \overset{150}{\cancel{240}}$

$= \lambda^2 + \lambda - 156 + \overset{150}{\cancel{240}} = \lambda^2 + \lambda \overset{-6}{\cancel{+84}}$

$(\lambda+3)(\lambda-2) \qquad \lambda = -3, 2$

b) $\lambda = -3 \quad (A-\lambda I)x \quad \begin{bmatrix} 15 & 15 \\ -10 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda = 2: \quad (A-2I)x \quad \begin{bmatrix} 10 & 15 \\ -10 & -15 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} \quad x = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$

c) $D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \quad P = \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix}$

d) $A^k = PD^kP^{-1} = \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 2^k & 0 \\ 0 & (-3)^k \end{bmatrix}\begin{bmatrix} +1 & +1 \\ -2 & -3 \end{bmatrix}$

$e^{At} = Pe^{Dt}P^{-1} = \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix}$

Q7 a) $L = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$

$$\begin{tikzcd} \mathbb{R}^2_T \arrow[r, "L"] & \mathbb{R}^2_T \\ \mathbb{R}^2_S \arrow[u, "P"] \arrow[r] & \mathbb{R}^2_S \arrow[u, "S"'] \end{tikzcd}$$

b) $P = \begin{bmatrix} 2 & 1 \\ -3 & -1 \end{bmatrix}$

c) $P^{-1}LP \quad \begin{bmatrix} -1 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ 11 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -5 & -3 \\ 19 & 10 \end{bmatrix}$

Q8 $A^{-1} = A^T \Rightarrow \frac{1}{\det(A)} = \det(A) \Rightarrow \det(A)^2 = 1 \Rightarrow \det(A) = \pm 1$

Q9 a) 4 rows, linearly dependent $\Rightarrow$ rank$(A) \leq 3$. $\Rightarrow$ dim col$(A) \leq 3$.

b) # pivots = col rank $\leq 3 \Rightarrow$ # free vars $\geq 2 \Rightarrow$ dim ker$(A) \geq 2$.

Q10 a) $Ax = 0$: $x_4 = 0$ $\quad x_3 - x_4 = 0 \Rightarrow x_3 = 0$ $\quad x_2 = t$

$x_1 - t + 0 = 0$ $\quad x_1 = t$ $\quad$ solutions: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} t \right\}$.

b) $Ax = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $\quad LUx = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $\quad$ solve $Ly = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$$\text{so } y = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$y_1 = 0$ $\quad -2y_1 + y_2 = 1 \Rightarrow y_2 = 1$ $\quad y_1 + 2y_2 + y_3 = 1 \Rightarrow y_3 = -1$

solve $Ux = y$ $\quad x_4 = -1$ $\quad x_3 - x_4 = 1$ $\quad x_3 = 0$ $\quad x_2 = t = 0$

$x_1 = 0$ $\quad$ so soln's are $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} t \right\}$.

c) $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \right\}$ $\quad$ d) $\{ [1\ -1\ 2\ 0], [0\ 0\ 1\ -1], [0\ 0\ 0\ 1] \}$.

Q11 a) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

b) $tr(A)$: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto a + d$. $\quad A = [a_{ij}]$ $\quad B = [b_{ij}]$.

sums: $tr(A+B) = a_{11} + a_{22} + b_{11} + b_{22} = tr(A) + tr(B)$. ✓

linear mult. $tr(cA) = ca_{11} + ca_{22} = c\,tr(A)$ ✓

c) $f(A) = A - \frac{1}{2} tr(A) I$.

sums: $f(A+B) = A+B - \frac{1}{2}\operatorname{tr}(A+B)I$

$= A+B-\frac{1}{2}(\operatorname{tr}(A)+\operatorname{tr}(B))I = \left(A-\frac{1}{2}\operatorname{tr}(A)\right) + \left(B-\frac{1}{2}\operatorname{tr}(B)I\right)$

$= f(A)+f(B)$. ✓

linear multi: $f(cA) = cA - \frac{1}{2}\operatorname{tr}(cA)I = cA - \frac{1}{2}c\operatorname{tr}(A)I$

$= c\left(A-\frac{1}{2}\operatorname{tr}(A)\right)I = cf(A)$ ✓

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \frac{1}{2}(a+d)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(a-d) & b \\ c & \frac{1}{2}(d-a) \end{bmatrix}$$

$$f = \begin{bmatrix} 1/2 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1/2 & 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ a \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(a-d) \\ b \\ c \\ \frac{1}{2}(d-a) \end{bmatrix}$$

$$\sim \begin{bmatrix} 1/2 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

← image has dim 3 w/ basis ~~spanned by~~

$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ ← $\operatorname{tr}(A)=0$ matrices.

kernel: $x_4 = t$ $x_3 = 0$ $x_2 = 0$ $\frac{1}{2}x_1 - \frac{1}{2}x_4 = 0$ $x_1 = t$.

~~span~~ basis $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ ← matrices of the form tI.