

3 things to change: function, differential, limits.

mnemonic: cancelling fractions $dx = \frac{dx}{du} du$

why does this work?

$$\int f(x) dx, x(u) \quad \text{set } F(x) = \int f(x) dx, \text{ so } F'(x) = f(x)$$

$$\int f(x) dx = F(x) \quad \begin{array}{c} \text{diff} \\ \text{wrt } u \end{array} \quad \frac{d}{du}(F(x(u))) = F'(x(u)) \cdot x'(u)$$

$$\int f(x) dx = \begin{array}{c} \text{integrate} \\ \text{wrt } u \end{array} = \int f(x(u)) x'(u) du = \int f(x(u)) \frac{dx}{du} du$$

useful fact: $\frac{du}{dx} = \frac{1}{\frac{dx}{du}}$

Examples

①

$$\int_0^1 e^{-7x} dx$$

$$\text{set } u = -7x \\ \frac{du}{dx} = -7$$

$$\int_0^{-7} e^u \frac{dx}{du} du = \int_0^{-7} e^u \cdot \frac{1}{-7} du = -\frac{1}{7} \int_0^{-7} e^u du = -\frac{1}{7} [e^u]_0^{-7} = -\frac{1}{7} (e^{-7} - 1)$$

② $\int_0^2 x^2 \sqrt{x^3+1} dx$, $\int_0^{\pi/4} \tan^3 x \sec^2 x dx$, $\int_0^1 \frac{x}{x^2+1} dx$.

§ 5.8 More integrals

recall: $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$ so $\int \frac{1}{x} dx = \ln|x| + c$

$$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x) + c$$

$$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1}(x) + c$$

$$\frac{d}{dx}(\sec^{-1}(x)) = \frac{1}{|x| \sqrt{x^2 - 1}}$$

$$\int \frac{1}{|x| \sqrt{x^2 - 1}} dx = \sec^{-1}(x) + C$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\int e^x dx = e^x + C$$

recall $b^x = e^{x \ln(b)}$

$$\int b^x dx = \int e^{x \ln(b)} dx = \frac{1}{\ln(b)} e^{x \ln(b)} + C$$

(sub $u = x \ln(b)$)

Examples

① $\int \frac{1}{9+x^2} dx = \frac{1}{9} \int \frac{1}{1+x^2/9} dx = \frac{1}{9} \int \frac{1}{1+(x/3)^2} dx$ sub $u = x/3 \dots$

② $\int \frac{1}{\sqrt{1-4x^2}} dx = \int \frac{1}{\sqrt{1-(2x)^2}} dx$ sub $u = 2x \dots$

③ $\int x(x+1)^9 dx$ ④ $\int \sqrt{4x-1} dx$ ⑤ $\int \sin^2 x \cos x dx$

⑥ $\int x \cos(x^2) dx$ ⑦ $\int x \sqrt{x^2-1} dx$ ⑧ $\int \frac{1}{x \ln(x)} dx$

⑨ $\int \frac{e^{2x} + e^{4x}}{e^x} dx$ ⑩ $\int 2^x dx$ ⑪ $\int x e^{x^2} dx$