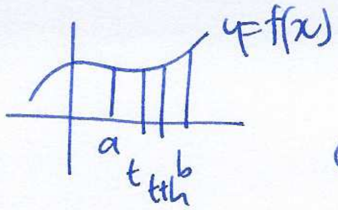
<u>intuition</u>



consider $\int_a^t f(x)\,dx$

Q: what is the rate of change wrt t?

$$\frac{d}{dt}\int_a^t f(x)\,dx = \lim_{h\to 0}\frac{\int_a^{t+h} f(x)\,dx - \int_a^t f(x)\,dx}{h} = \lim_{h\to 0}\frac{1}{h}\int_t^{t+h} f(x)\,dx$$

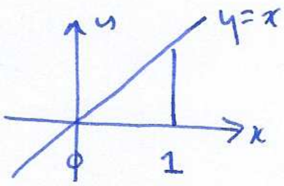
$$\approx \frac{\text{area of rectangle}}{h} = \frac{f(t)\times h}{h} = f(t)$$

i.e. $\int_a^t f(x)\,dx$ is an antiderivative for $f(x)$, so $\int_a^t f(x)\,dx = F(t)+c$

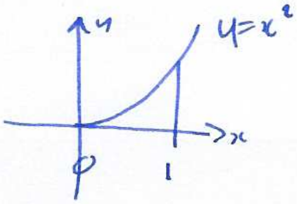
Q: what is the constant? $t=a$: $\int_a^a f(x)\,dx = 0 = F(a)+c \Rightarrow c=-F(a)$

so $\int_a^t f(x)\,dx = F(t)-F(a)$ $\square$

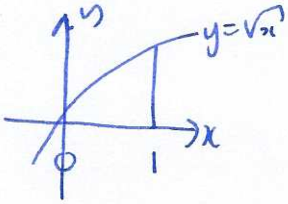
<u>Examples</u>



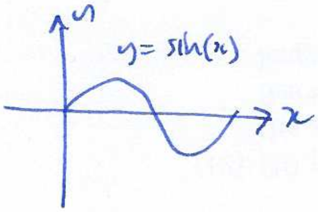
$$\int_0^1 x\,dx = \left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{1}{2}-0 = \tfrac{1}{2}$$



$$\int_0^1 x^2\,dx = \left[\tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{3}-0 = \tfrac{1}{3}$$



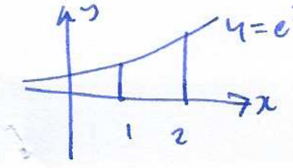
$$\int_0^1 x^{1/2}\,dx = \left[\tfrac{2}{3}x^{3/2}\right]_0^1 = \tfrac{2}{3}-0 = \tfrac{2}{3}$$



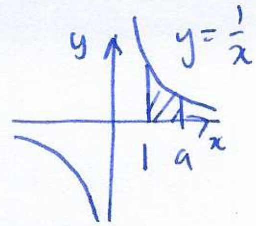
$$\int_0^\pi \sin x\,dx = \left[-\cos x\right]_0^\pi = -\cos(\pi)+\cos(0) = -(-1)+1 = 2$$

$$\int_0^{2\pi} \sin x\,dx = \left[-\cos x\right]_0^{2\pi} = -\cos(2\pi)+\cos(0) = -1+1 = 0$$

Q: why?



$$\int_1^2 e^x\,dx = \left[e^x\right]_1^2 = e^2-e$$



$$\int_1^a \frac{1}{x}\,dx = \Big[\ln|x|\Big]_1^a = \ln|a| - \ln(1) = \ln(a)$$

Observations

① choice of antiderivative doesn't matter, let $F(x)$ and $F(x)+c$ be antiderivatives for $f(x)$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$= F(b)+c - (F(a)+c) = F(b) - F(a)$$

② $\int_a^t f(x)\,dx$ is a function of t! (and a, f) but not x (called a dummy variable) i.e. $\int_a^t f(x)\,dx = \int_a^t f(y)\,dy$. If you want a function of x write $\int_a^x f(t)\,dt$

§ 5.4 Fundamental theorem of calculus II

Thm (FTC ②) Let $f(x)$ be a continuous function on $[a,b]$, then $A(x) = \int_a^x f(t)\,dt$ is an antiderivative for $f(x)$, i.e. $A'(x) = f(x) = \frac{dA}{dx}$, so $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$. Furthermore, $A(a) = 0$ $\square$.

Example $\frac{d}{dx}\int_0^{x^2} \cos(t)\,dt$.

§ 5.6 Substitution / change of variable

"reverse chain rule for integration"

recall: chain rule: $\frac{d}{dx}\big(f(g(x))\big) = f'(g(x)).g'(x)$

substitution / change of variable

$\int f(x)\,dx$, $x(u)$

$$\int_{x=a}^{x=b} f(x)\,dx = \int_{u=x^{-1}(a)}^{u=x^{-1}(b)} f(x(u))\frac{dx}{du}\,du$$