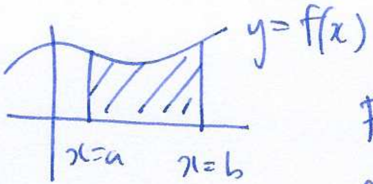
§5.2 Definite integral

want: $\int_a^b f(x)\,dx =$ area under the curve $y=f(x)$ between $x=a$ and $x=b$.



Formal defn: Riemann sum $R(f,P,c)$

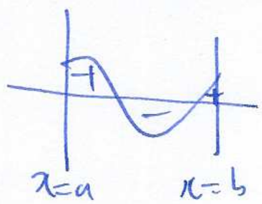
$$\int_a^b f(x)\,dx = \lim_{\|\Delta x_i\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

P partition $a = x_0 < x_1 < \cdots < x_n = b$

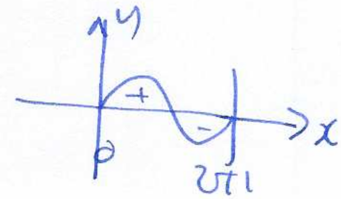
$\Delta x_i = x_i - x_{i-1}$

$c_i \in [x_{i-1}, x_i]$.

Note: signed area!

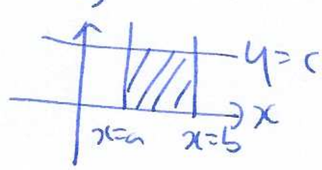


so $\int_0^{2\pi} \sin(x)\,dx = 0$



useful properties

$\int_a^b c\,dx = c(b-a)$



sums

$\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

constant multiple

$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$.

adjacent intervals: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

0-length interval $\int_a^a f(x)\,dx = 0$

reversing limits: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

comparisons: $f(x) \le g(x)$ then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

§5.3 Antiderivatives

Defn A function $F(x)$ is an anti-derivative for $f(x)$ if $F'(x) = f(x)$.

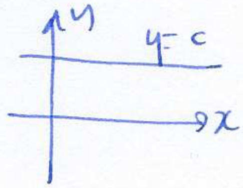
General antiderivative ↑ example $f(x) = x^2$, $F(x) = \frac{1}{3}x^3$. note! $\frac{1}{3}x^3+1$ also works.

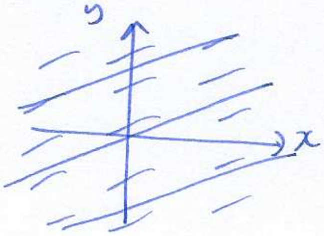
Thm Let $F(x)$ be an antiderivative for $f(x)$, then any other antiderivative has the form $F(x)+c$ for some $c \in \mathbb{R}$.

Proof suppose $F(x)$ and $G(x)$ are antiderivatives for $f(x)$, then $F'(x) = G'(x) = f(x)$, so $(F(x)-G(x))' = f(x) - f(x) = 0$, so $F(x) - G(x) = c$ constant □

Picture $f(x)$ gives the slope function for $F(x)$.

Example $f(x) = c$





$F'(x) = c$

$f(x) = cx + d$.

~~Examples~~ Notation $\int f(x)\,dx$ means general antiderivative $F(x) + c$.

Example $\int x^2 + \frac{1}{x} + \sin x \, dx = \frac{1}{3}x^3 + \ln|x| - \cos x + c$.

Observation: every rule for differentiation gives a rule for integration.

Warning: no easy analog of product / quotient / chain rule!

Alternate view we can think of finding the indefinite integral as finding a function given its slopes, i.e. its derivative. This is an example of solving a differential equation $\frac{dy}{dx} = f(x)$. In general, there is a family of solutions $F(x) + c$, but if we know the value of the solution we want at $x = 0$ (sometimes called an initial condition) then this picks out a particular solution.

Example: motion under gravity, acceleration: $a(t) = x''(t) = -g$ constant

velocity: $v(t) = x'(t) = -\cancel{\tfrac{1}{2}}\,g t + c$

if $v(0) = v_0$ initial velocity, then $v(t) = -gt + v_0$

position: $x(t) = -\frac{1}{2} g t^2 + v_0 t + c$

if $x(0) = x_0$ initial position, then $x(t) = -\frac{1}{2} g(t) + v_0 t + x_0$.

§5.4 Fundamental theorem of calculus

Thm (FTC ①) suppose $f(x)$ is continuous on $[a,b]$ and $F(x)$ is an anti-derivative for $f(x)$. Then $\int_a^b f(x)\,dx = F(b) - F(a)$.