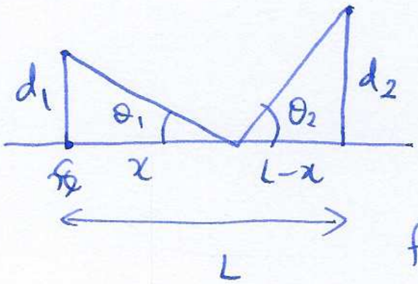
$A'(r) = 4\pi r + -\frac{2}{r^2}$ solve $A'(r) = 0$: $r^3 = \frac{1}{2\pi}$ $r = \sqrt[3]{1/2\pi}$.

<u>Example</u> a ball bounces off a wall. show that the path which minimizes distance has equal angles of incidence and reflection.



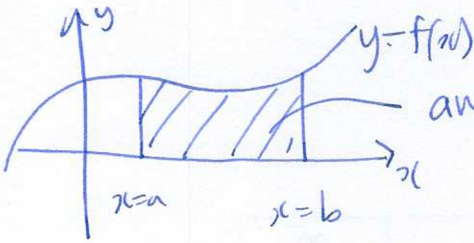
$f(x)$ - distance = $\sqrt{d_1^2 + x^2} + \sqrt{d_2^2 + (L-x)^2}$

$f'(x) = \frac{1}{2}(d_1^2 + x^2)^{-1/2} \cdot 2x + \frac{1}{2}(d_2^2 + (L-x)^2)^{-1/2} \cdot 2(L-x)(-1)$

$f'(x) = 0$: $\frac{x}{\sqrt{d_1^2 + x^2}} = \frac{L-x}{\sqrt{d_2^2 + (L-x)^2}} \iff \cos\theta_1 = \cos\theta_2$

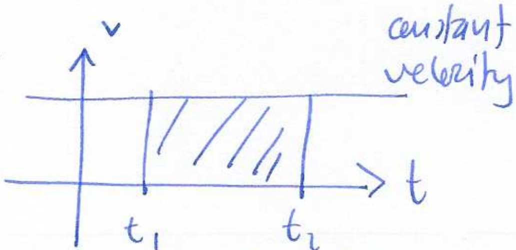
$\Rightarrow \theta_1 = \theta_2$

§5.1 Approximating area



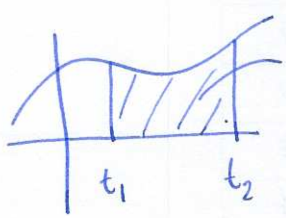
area under the graph

<u>Example</u>



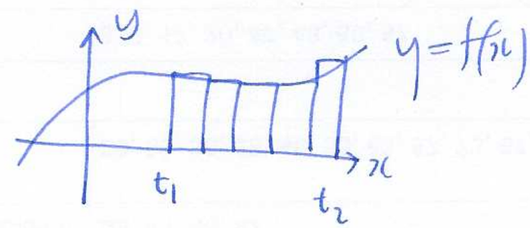
distance travelled = velocity × time
= area under the graph

non-constant velocity :

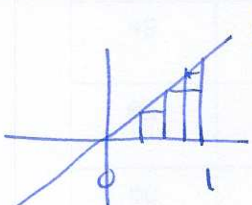


fact: distance travelled = area under the graph

finding the area : approximate by rectangles



<u>Example</u>



find area under graph of $y = x$ between $x = 0$ and $x = 1$
(answer $= \frac{1}{2}$)

approximate with 4 rectangles : area ≈ sum of $\underbrace{\text{area of rectangles}}_{\text{width} \times \text{height}}$

$\approx \frac{1}{4}f(0) + \frac{1}{4}f(\frac{1}{4}) + \frac{1}{4}f(\frac{1}{2}) + \frac{1}{4}f(\frac{3}{4})$

$= \frac{1}{4}\left(f(0) + f(\frac{1}{4}) + f(\frac{2}{4}) + f(\frac{3}{4})\right) = \frac{1}{4}\sum_{i=0}^{3} f(\frac{i}{4})$

$= \frac{1}{4}\left(0 + \frac{1}{4} + \frac{2}{4} + \frac{3}{4}\right) = \frac{1}{4}\frac{6}{4} = \frac{3}{8} \approx 0.375$

approximate with n rectangles:

$$f(0)\frac{1}{n} + f\left(\frac{1}{n}\right)\frac{1}{n} + f\left(\frac{2}{n}\right)\frac{1}{n} + \cdots + f\left(\frac{n-1}{n}\right)\frac{1}{n} = \sum_{i=0}^{n-1} \frac{1}{n} f\left(\frac{i}{n}\right)$$

$$= \frac{1}{n}\left(f(0) + \cdots + f\left(\frac{n-1}{n}\right)\right) \qquad = \sum_{i=0}^{n-1} \frac{1}{n}\cdot\frac{i}{n}$$

$$= \frac{1}{n}\left(0 + \frac{1}{n} + \frac{2}{n} + \cdots + \frac{n-1}{n}\right) = \frac{1}{n^2}\sum_{i=0}^{n-1} i$$

$$= \frac{1}{n^2}\left(1 + 2 + 3 + \cdots + n-1\right)$$

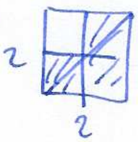
claim: $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$

Proof ① induction: ~~as~~ assume true for k: $S_k = 1 + 2 + \cdots + k = \frac{1}{2}k(k+1)$

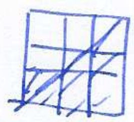
$$S_{k+1} = \underbrace{1 + 2 + \cdots + k}_{S_k} + (k+1) = \frac{1}{2}k(k+1) + (k+1) = (k+1)\left(\frac{1}{2}k + 1\right) = \frac{1}{2}(k+1)(k+2) \checkmark.$$

check base case: $k = 1$ $\quad S_1 = \frac{1}{2}1.2 = 1 \checkmark$. $\square$.

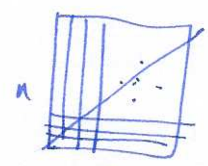
②



$\frac{1}{2}(2)^2 + 2\frac{1}{2}$



$\frac{1}{2}(3)^2 + \frac{1}{2}(3)$



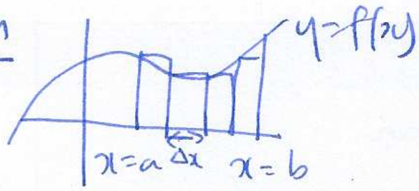
$\frac{1}{2}n^2 + \frac{1}{2}n$.

$= \frac{1}{2}n(n+1)$.

so approximate area is $\frac{1}{n^2}\left(1 + 2 + \cdots + (n-1)\right) = \frac{1}{n^2}\frac{1}{2}(n-1)n = \frac{1}{2}\frac{n^2 - n}{n^2} = \frac{1}{2}\left(1 - \frac{1}{n}\right)$

$\to \frac{1}{2}$ as $n \to \infty$.

Notation



N rectangles of equal width

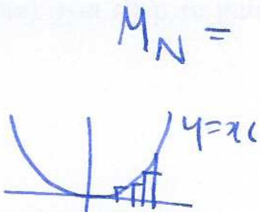
then $\Delta x = \frac{b-a}{N}$

left endpoint rectangles $\quad L_N = \sum_{i=0}^{N-1} f(a + i\Delta x)\,\Delta x$

right endpoint rectangles $\quad R_N = \sum_{i=1}^{N} f(a + i\Delta x)\,\Delta x$

midpoint rectangles $\quad M_N = \sum_{i=1}^{N} f\left(a + \left(i - \frac{1}{2}\right)\Delta x\right)\Delta x$

Q: what about $y = x^2$



need to find $1^2 + 2^2 + \cdots n^2 = \frac{1}{6}n(n+1)(2n+1) \cdots$

need better way...