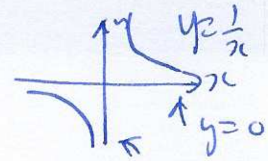
Asymptotes

Example $y=\frac{1}{x}$



Defn $x=c$ is a vertical asymptote if $\lim_{x\to c\pm} f(x) = \pm\infty$

$y=c$ is a horizontal asymptote if $\lim_{x\to\pm\infty} f(x) = c$

Observation: rational functions $\frac{p(x)}{q(x)}$ have horizontal asymptotes if $\deg p \leq \deg q$

Example $f(x) = \frac{x^2+x+1}{3x^2+2} \sim \frac{x^2}{3x^2} \to \frac{1}{3}$ as $x\to\infty$

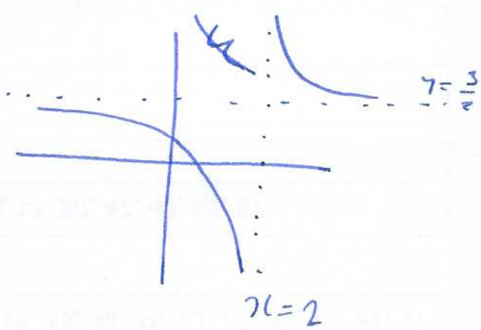
Example sketch graph of $f(x) = \frac{3x+2}{2x-4}$

① find vertical asymptotes $\leftrightarrow$ denominator zero : $2x-4=0$, $x=2$

② find $f'(x) = \frac{(2x-4).3-(3x+2).2}{(2x-4)^2} = \frac{-16}{(2x-4)^2} = \frac{-4}{(x-2)^2}$ ← always $-$ve

f decreasing, no critical points except vertical asymptote at $x=2$

③ $f''(x) = \frac{8}{(x-2)^3}$ $\quad$ +ve $\quad x>2$ $\quad$ concave up
$\quad$ $-$ve $\quad x<2$ $\quad$ concave down



④ horizontal asymptotes : $\frac{3x+2}{2x-4} = \frac{3+2/x}{2-4/x} \to \frac{3}{2}$

behaviour near asymptote:

		$-2/3$		2	
$3x+2$	$-$		$+$		$+$
$2x-4$	$-$		$-$		$+$
$f(x)$	$+$		$-$		$+$

Example sketch graph of $f(x) = \frac{x}{\sqrt{x^2+1}} = x(x^2+1)^{-1/2}$

① vertical asymptotes: none

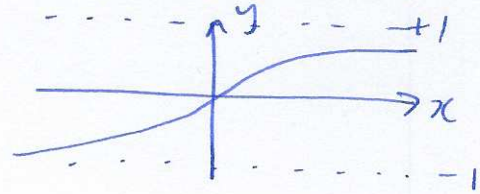
② find $f'(x) = \frac{\sqrt{x^2+1}.1 - x.\frac{1}{2}(x^2+1)^{-1/2}.2x}{x^2+1} = \frac{x^2+1-x^2}{(x^2+1)^{3/2}} = (x^2+1)^{3/2} > 0$

$\Rightarrow$ f increasing, no critical points

③ $f''(x) = -\frac{3}{2}(x^2+1)^{-\frac{5}{2}}\, 2x$ point of inflection at $x=0$ +ve for $x<0$
−ve for $x>0$

④ horizontal asymptotes $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} \overset{l'H}{=} \lim_{x\to\infty} \frac{1}{\frac{1}{2}(x^2+1)^{-1/2}\cdot 2x} = \lim_{x\to\infty} \sqrt{1+1/x^2} = 1$

$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{-x}{\sqrt{x^2+1}} = -1$

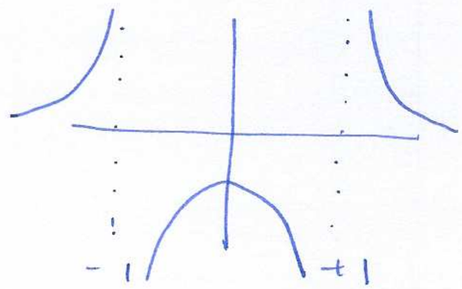


Example $f(x) = \frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)}$

① vertical asymptotes at $x = \pm 1$

② $f'(x) = -(x^2-1)^{-2}\cdot 2x$, critical point at $x=0$

③ $f''(x) = \frac{6x^2+2}{(x^2-1)^3}$



④ etc.

§4.7 Optimization

Example a piece of wire of length L is bent into a rectangle. What's the largest possible area?

area $A = xy$
length $L = 2x+2y$ $\Big\}$ $y = \frac{L}{2} - x$

$A = x\left(\frac{L}{2} - x\right) = \frac{L}{2}x - x^2$

$A'(x) = \frac{L}{2} - 2x$ critical point $x = \frac{L}{4}$ (local max)

so max area of $\frac{L^2}{16}$ occurs when $x = y = \frac{L}{4}$ (square)

Example what shape of cylindrical can minimizes surface area, if you want total volume to be $1\,ft^3$?

$V = \pi r^2 h = 1 \Rightarrow h = \frac{1}{\pi r^2}$

$A = 2\pi r^2 + 2\pi r h$

$A = 2\pi r^2 + 2\pi r \frac{1}{\pi r^2} = 2\pi r^2 + \frac{2}{r}$