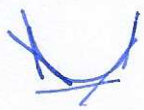
Q: how do the slopes change?



concave up

concave down

$\Leftrightarrow$ slopes increasing $\Leftrightarrow$ slopes decreasing

$\Leftrightarrow$ $f''(x) > 0$ $\Leftrightarrow$ $f''(x) < 0$

Defn Let $f(x)$ be differentiable on an interval (a,b), then

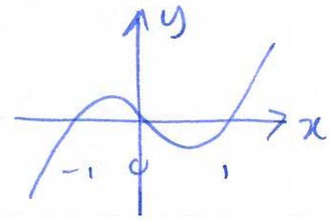
$f(x)$ is concave up $\Leftrightarrow$ $f''(x) > 0$

$f(x)$ is concave down $\Leftrightarrow$ $f''(x) < 0$

Defn An inflection point is where the graph changes from concave up to concave down, or vice versa.

Note: inflection point $\Rightarrow$ $f''(x) = 0$

$\not\Leftarrow$ $f''(x) = 0$



example $f(x) = x^3 - x = x(x^2-1) = x(x-1)(x+1)$

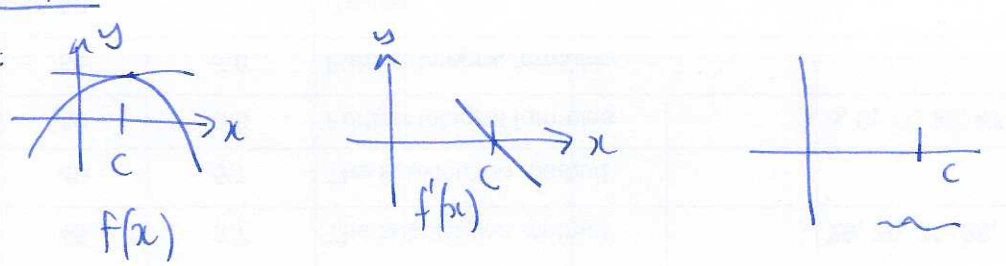
$f'(x) = 3x^2 - 1$

$f''(x) = 6x$

$\left.\begin{array}{ll} f''(x) > 0 & \text{for } x > 0 \\ f''(x) < 0 & \text{for } x < 0 \end{array}\right\}$ inflection pt at $x = 0$

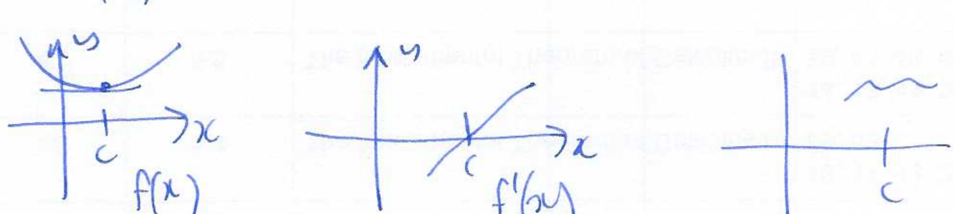
Second derivative

local max:



$f''(x) < 0$

local min:



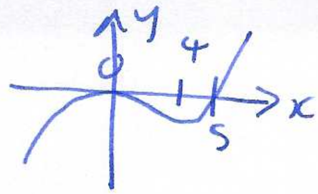
$f''(x) > 0$

Thm suppose $f(x)$ is differentiable ~~at~~ and c is a critical point

If $f''(c) > 0 \Rightarrow c$ is a local min

$f''(c) < 0 \Rightarrow c$ is a local max

$f''(c) = 0 \Rightarrow$ NO INFORMATION (may be local/max/min/neither)



Example $f(x) = x^5 - 5x^4 = x^4(x-5)$

$f'(x) = 5x^4 - 20x^3 = 5x^3(x-4)$

$f''(x) = 20x^3 - 60x^2$

critical points : $f'(x) = 0 \Rightarrow x = 0, 4$

2nd derivative test : $x=0$ $f''(0) = 0$ no information

$x = 4$ $f''(4) = 320 > 0$ local min

at $x=0$ use first derivative test :

		0	
$(x-4)$	$-$		$-$
x^3	$-$	0	$+$
$f'(x)$	$+$		$-$

$\Rightarrow$ local max

§4.5 L'Hôpitals rule

Thm Suppose $f(x)$ and $g(x)$ are differentiable, and $f(a) = g(a) = 0$ or $f(a) = g(a) = \pm\infty$

then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, provided this limit exists.

warning ① this is not the quotient rule!

② $\frac{f(a)}{g(a)}$ must be indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, not $\frac{1}{0}$, i.e. can't use l'H on $\lim_{x \to 0} \frac{1}{x}$

Examples ① $\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1} \frac{3x^2}{1} = 3$

② $\lim_{x \to 1} \frac{x^{100} - 1}{x - 1} = \lim_{x \to 1} \frac{100x^{99}}{1} = 100$

③ $\lim_{x \to \frac{\pi}{2}} \frac{\cos^2 x}{\sin x - 1} = \lim_{x \to \frac{\pi}{2}} \frac{-2\cos x \sin x}{\cos x} = \lim_{x \to \frac{\pi}{2}} -2\sin x = -2$

④ $\lim_{x \to 0^+} x\ln(x) = \lim_{x \to 0^+} \frac{\ln(x)}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} -x = 0$

⑤ $\lim_{x \to 0} \frac{e^x - x - 1}{\cos x - 1} = \lim_{x \to 0} \frac{e^x - 1}{-\sin x} = \lim_{x \to 0} \frac{e^x}{-\cos x} = -1$

(6) $\lim\limits_{x\to 0} \frac{1}{\sin x} - \frac{1}{x} = \lim\limits_{x\to 0} \frac{x-\sin x}{x\sin x} = \lim\limits_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x} = \lim\limits_{x\to 0} \frac{\sin x}{\cos x + \cos x - x\sin x} = 0$

(7) $\lim\limits_{x\to 0^+} x^x = \lim\limits_{x\to 0} e^{x\ln x}$ note: e^x continuous so $= e^{\lim\limits_{x\to 0^+} x\ln x} = e^0 = 1$

Comparing growth rates of functions

Q: which grows faster $(\ln(x))^2$ or $\sqrt{x}$?

$\lim\limits_{x\to\infty} \frac{\sqrt{x}}{(\ln(x))^2} = \lim\limits_{x\to\infty} \frac{\frac{1}{2}x^{-1/2}}{2\ln(x)\cdot\frac{1}{x}} = \lim\limits_{x\to\infty} \frac{\sqrt{x}}{4\ln(x)} = \lim\limits_{x\to\infty} \frac{\frac{1}{2}x^{-1/2}}{4/x} = \lim\limits_{x\to\infty} \frac{\sqrt{x}}{8} = \infty$

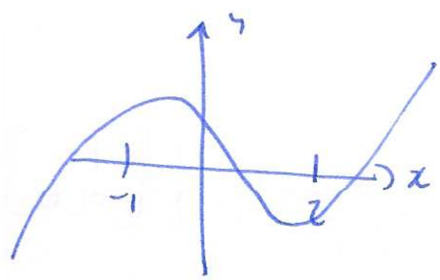
so $\sqrt{x}$ grows faster.

Thm e^x grows faster than any polynomial

Proof $\lim\limits_{x\to\infty} \frac{e^x}{x^n} = \lim\limits_{x\to\infty} \frac{e^x}{nx^{n-1}} = \cdots = \lim\limits_{x\to\infty} \frac{e^x}{n!} = \infty$ $\square$.

§ 4.6 Graph sketching and asymptotes

Example sketch graph of $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 3$



helpful info:
- critical points $f'(x) = x^2 - x - 2$
  $(x-2)(x+1)$
- sign of $f'(x)$
- sign of $f''(x)$ $f''(x) = 2x - 1$

critical points at $x = -1, 2$

$f''(-1) = -3 < 0$ local max

$f''(2) = 3 > 0$ local min

Example $f(x) = (4x - x^2)e^x$ critical points $x = 1 \pm \sqrt{5}$

$f'(x) = (4 + 2x - x^2)e^x$

$f''(x) = (6 - x^2)e^x$

$1+\sqrt{5}$ local max

$1-\sqrt{5}$ local min.