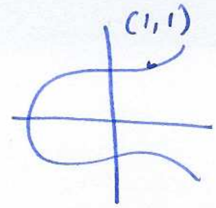
§3.8 Implicit differentiation

suppose $y^4 + xy = x^3 - x + 2$



can't solve for y explicitly

$(y(x))^4 + x\,y(x) = x^3 - x + 2$ ← diff wrt x using chain rule on $y(x)$.

$4(y(x))^3 \cdot y'(x) + y(x) + xy'(x) = 3x^2 - 1$

$y'(x)(4y^3 + x) = 3x^2 - 1 - y$ $\qquad y'(x) = \dfrac{3x^2-1-y}{4y^3+x}$

Example $x^2+y^2=1$ $\quad 2x + 2yy' = 0$ $\quad y' = -x/y$

Application : derivatives of inverse functions.

Example $y = \ln(x)$.

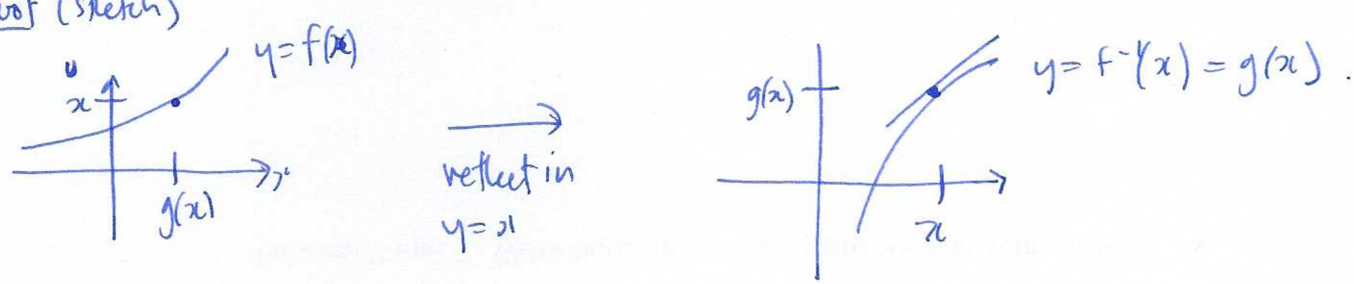
$e^y = x$

$e^y \cdot y' = 1$ $\qquad y' = e^{-y} = e^{-\ln(x)} = \dfrac{1}{x}$

Thm If $f(x)$ is differentiable, one-to-one, with inverse $f^{-1}(x) = g(x)$

then $g'(x) = \dfrac{1}{f'(g(x))}$ as long as $f'(g(x)) \neq 0$

Proof (sketch)



$\dfrac{1}{\text{slope}}$ at $g(x)$, i.e. $\dfrac{1}{f'(g(x))}$ $\quad \leftrightarrow \quad$ slope at x: $g'(x)$

$\square$.

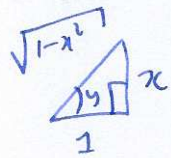
Application : derivatives of inverse trig functions.

Thm $\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}\left(\cos^{-1}(x)\right) = \frac{-1}{\sqrt{1-x^2}}$

Proof (of $\sin^{-1}(x)$):

$$y = \sin^{-1}(x)$$

$$\sin(y) = x$$



$$\cos(y)\,y' = 1$$

$$y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

Thm $\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}$ $\frac{d}{dx}\left(\cot^{-1}(x)\right) = \frac{-1}{1+x^2}$

$\frac{d}{dx}\left(\sec^{-1}(x)\right) = \frac{1}{|x|\sqrt{x^2-1}}$ $\frac{d}{dx}\left(\csc^{-1}(x)\right) = \frac{-1}{|x|\sqrt{x^2-1}}$

Proof (of $\tan^{-1}(x)$)

$$y = \tan^{-1}(x)$$

$$\tan(y) = x$$

$$\sec^2(y)\,y' = 1$$

$$y' = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2} \quad \square.$$

Example $\frac{d}{dx}\left(\tan^{-1}(e^{2x})\right) = \frac{1}{1+(e^{2x})^2} \cdot e^{2x} \cdot 2$

§3.9 Derivatives of exponentials and logs

recall $f(x) = b^x$ then $f'(x) = m_b b^x$

$f(x) = e^x$ then $f'(x) = e^x$

Thm $f(x) = b^x$ then $f'(x) = \ln(b)\, b^x$

Proof $f(x) = b^x = e^{\ln(b)x}$ $f'(x) = e^{\ln(b).x} . \ln(b) = b^x . \ln(b)$ $\square$

Example $f(x) = 3^{4x} = (3^4)^x$. $f'(x) = \ln(3^4)(3^4)^x = 4\ln(3)3^{4x}$.

Thm $f(x) = \ln(x)$ then $f'(x) = \frac{1}{x}$

Proof $y = \ln(x)$ $\quad$ $y' = \frac{1}{e^y} = \frac{1}{x}$ $\square$

$e^y = x$

$e^y \cdot y' = 1$

Example ① $f(x) = \log_b(x) = \frac{\ln(x)}{\ln(b)}$ $\quad$ $f'(x) = \frac{1}{x\ln(b)}$

② $f(x) = x\ln(x)$ $\quad$ $f'(x) = \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1$

Hyperbolic trig functions

$\sinh(x) = \frac{e^x - e^{-x}}{2}$ $\quad$ $\cosh(x) = \frac{e^x + e^{-x}}{2}$

$\frac{d}{dx}(\sinh(x)) = \cosh(x)$ $\quad$ $\frac{d}{dx}(\cosh(x)) = \sinh(x)$ $\quad$ note $\cosh^2 x - \sinh^2 x = 1$

check: $(\sinh(x))' = \frac{1}{2}(e^x - e^{-x})' = \frac{1}{2}(e^x + e^{-x}) = \cosh(x)$

$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$ so $\frac{d}{dx}(\tanh(x)) = \frac{1}{\cosh^2 x}$ etc.

inverse functions: $\frac{d}{dx}(\sinh^{-1}(x)) = \frac{1}{\sqrt{x^2+1}}$ $\quad$ $\frac{d}{dx}(\cosh^{-1}(x)) = \frac{1}{\sqrt{x^2-1}}$

$\frac{d}{dx}(\tanh^{-1}(x)) = \frac{1}{1-x^2}$

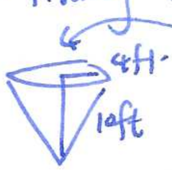
§3.10 Related rates

Example ① Balloon  $V = \frac{4}{3}\pi r^3$ $\quad$ inflate balloon: $V(t) = \frac{4}{3}\pi r(t)^3$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ so if $\frac{dV}{dt} = 1\,ft^3/min$ and $r = 1ft$: $\quad 1 = 4\pi \cdot 1 \cdot \frac{dr}{dt}$

so $\frac{dr}{dt} = \frac{1}{4\pi}$ ft/min

② filling a conical tank



$10\,ft^3/min$ $\quad$ Q: how fast is the water rising when $h = 3ft$?

water in $\frac{dV}{dt} = 10ft^3/min$ $\quad$ volume of cone: $V = \frac{1}{3}\pi h r^3$