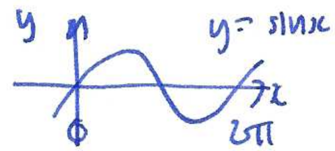
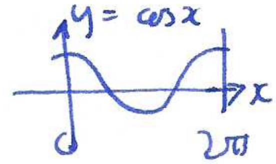
$$= \lim_{h\to 0} \frac{\sin x \cosh + \cos x \sin h - \sin x}{h} = \lim_{h\to 0} \sin x \cdot \frac{\cos h - 1}{h} + \lim_{h \to 0} \cos x \frac{\sin h}{h}$$

$$= \sin x \cdot \underbrace{\lim_{h\to 0} \frac{\cos(h)-1}{h}}_{0} + \cos x \underbrace{\lim_{h\to 0}\frac{\sin h}{h}}_{1} = \cos x . \square$$

Q: can this be right?





Example $f(x) = x \sin x$

$f'(x) = x\cos x + \sin x$

Thm $\frac{d}{dx}(\tan x) = \sec^2 x$ $\qquad \frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$ $\qquad \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x .$

Proof (of $\tan x$) $\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x . \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x .$ $\square$

Example $\frac{d}{dx}(e^x \cos x) = e^x . -\sin x + e^x \cos x .$

§ 3.7 Chain rule

Composition of functions : $f(g(x)) = (f\circ g)(x)$.

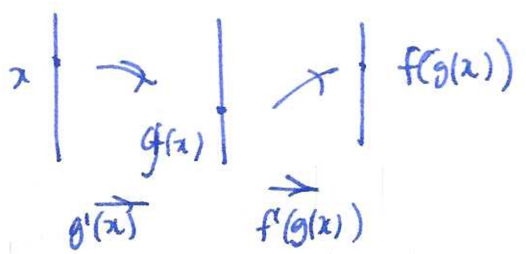
Examples e^{4x}, $\sin^2(x)$, etc. ...

Thm If f, g differentiable functions, then $f\circ g$ is differentiable, and

$(f(g(x)))' = f'(g(x)) . f'(x)$.

mnemonic : $(f(g(x)))' =$ outside'(inside) . inside'

note: $f(g(x))$ $\qquad \mathbb{R} \xrightarrow{g} \mathbb{R} \xrightarrow{f} \mathbb{R}$



Examples ① $e^{4x} = f(g(x))$ where $f(x) = e^x$ $\quad g(x) = 4x$

$f'(x) = e^x \quad g'(x) = 4$

so $(e^{4x})' = f'(g(x)) \cdot g'(x) = e^{4x} \cdot 4.$

② $\sin^2(x) = f(g(x))$, where $f(x) = x^2 \quad g(x) = \sin x$

$f'(x) = 2x \quad g'(x) = \cos x$

so $(\sin^2(x))' = 2 \sin x \cos x.$

③ $\sqrt{x^3+1}$, etc.

Alternate notation $f(g(x)) \leftrightarrow f(u) \quad u = g(x)$

$\frac{df}{dx} = f'(u)\frac{du}{dx} = \frac{df}{du}\frac{du}{dx}$ $\quad \boxed{\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}}$ mnemonic: "cancelling fractions."

Examples $\cos(x^2)$, $e^{\sqrt{x}}$, $\sin\left(\frac{\pi x}{180}\right)$, $\sqrt{x+\sqrt{x^2+1}}$.

Proof (of chain rule)

$[f(g(x))]' = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}$ [answer should be $f'(g(x)) \cdot g'(x)$].

write this as $\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \frac{g(x+h) - g(x)}{h}$

set $k = g(x+h) - g(x)$. As g is cts, $h \to 0 \Rightarrow k \Rightarrow 0$

so $= \lim_{k\to 0} \frac{f(g(x)+k) - f(g(x))}{k} \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = f'(g(x)) \cdot g'(x)$ □

Examples $\frac{d}{dx}\left(g(x)^n\right) = n(g(x))^{n-1} \cdot g'(x).$

$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)} \cdot g'(x)$

$\frac{d}{dx}\left(f(ax+b)\right) = a f'(ax+b).$