warning: this rule works for polynomials only, not exponentials.

$f(x) = x^{100}$ polynomial $\quad f'(x) = 100x^{99}$ $\quad f(x) = 2^x$ not polynomial.

other useful rules

Thm (linearity) If f and g are differentiable functions, then:

- $f+g$ is differentiable with $(f+g)' = f' + g'$ $\leftrightarrow$ $\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$
- k constant, $(kf)' = kf'$ $\leftrightarrow$ $\frac{d}{dx}(kf) = k\frac{df}{dx}$

Proof (follows from limit laws)

$$(f+g)'(x) = \lim_{h\to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = f'(x) + g'(x).$$

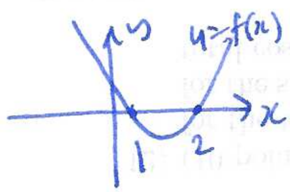
$$(kf)'(x) = \lim_{h\to 0} \frac{kf(x+h) - kf(x)}{h} = k\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = kf'(x) \quad \square$$

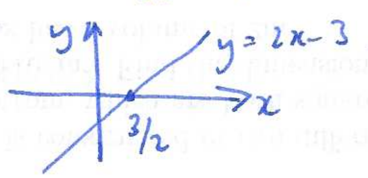
Example $f(x) = x^2 - 3x + 2$ find $f'(x)$

$$\frac{df}{dx} = \frac{d}{dx}(x^2 - 3x + 2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(-3x) + \frac{d}{dx}(2)$$

$$= 2x - 3\frac{d}{dx}(x) + 0 = 2x - 3.$$

graphs





Derivative of e^x

consider $f(x) = b^x$, $b > 0$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{b^{x+h}-b^x}{h} = \lim_{h\to 0} b^x \cdot \frac{b^h-1}{h}$$

$$= b^x \underbrace{\lim_{h\to 0} \frac{b^h-1}{h}}$$ doesn't depend on x! assume this limit exists and call it m_b

we have shown: for exponential functions, the derivative is <u>proportional</u> to the value of the original functions, i.e. if $f(x) = b^x$, $f'(x) = m_b b^x$, in particular the slope at $x=0$ is m_b.

<u>recall</u> e is defined to be the special number s.t. the slope of e^x at $x=0$ is equal to 1, therefore if $\boxed{f(x) = e^x, \text{ then } f'(x) = e^x}$ $\frac{d}{dx}(e^x) = e^x$.

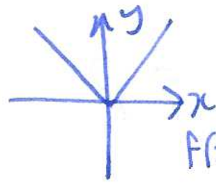
<u>Example</u> $\frac{d}{dx}(7e^x + 8x^2) = 7e^x + 16x$.

<u>observation</u>: this shows that e^x is <u>not</u> a polynomial.

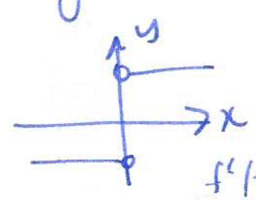
$\frac{d^n}{dx^n}(p(x))$ ← degree goes down, eventually zero

<u>Thm</u> Differentiable $\Rightarrow$ continuous (warning: continuous $\not\Rightarrow$ differentiable)

<u>Example</u> $f(x) = |x|$ continuous



$f(x)$



$f'(x)$ <u>not</u> continuous.

<u>claim</u>: $f(x) = |x|$ not differentiable at $x=0$.

<u>check</u>: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{|0+h|-|0|}{h} = \lim_{h\to 0} \frac{|h|}{h}$

$\lim_{h\to 0+} \frac{|h|}{h} = +1$ $\lim_{h\to 0-} \frac{|h|}{h} = -1 \Rightarrow \lim_{h\to 0} \frac{|h|}{h}$ DNE. $\square$.

<u>local picture</u> if $f(x)$ is differentiable at $x=c$, then if you look close enough, the graph looks close to a straight line

<u>Proof</u> (differentiable $\Rightarrow$ cts) $f(x)$ differentiable at $x=c$ means $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ exists. (want to show: $\lim_{x\to c} f(x) = f(c)$)

consider $f(c+h)-f(c) = h \cdot \frac{f(c+h)-f(c)}{h}$ so $\lim_{h\to 0} f(c+h)-f(c)$

$= \lim_{h\to 0} h \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = 0 \cdot f'(c) = 0$ $\square$.

§3.3 Product and quotient rules

new functions from old : $f(x)g(x)$ product $\quad \frac{f(x)}{g(x)}$ quotient.

Thm (product rule) $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

$$(fg)' = f'g + fg'$$

$$\frac{d}{dx}(fg) = \frac{d}{dx}f \cdot g + f\frac{dg}{dx}$$

warning $(fg)' \neq f'g'$!!

Examples ① $\frac{d}{dx}(x^2) = \frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x) \cdot x + x \cdot \frac{d}{dx}(x) = x + x = 2x$

② $\frac{d}{dx}\left(3x^2(x^2+1)\right) = (3x^2)'(x^2+1) + (3x^2)(x^2+1)' = 6x(x^2+1) + 3x^2 \cdot 2x$

③ $\frac{d}{dx}(x^2e^x) = \frac{d}{dx}(x^2)e^x + x^2\frac{d}{dx}(e^x) = 2xe^x + x^2e^x$

Proof (of product rule) (assume f, g both differentiable at x)

$(fg)'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$

$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$

$= \lim_{h\to 0} f(x+h)\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \cdot g(x)$

$= \lim_{h\to 0} f(x+h) \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) g(x)$

$= f(x)g'(x) + f'(x)g(x)$ $\square$.

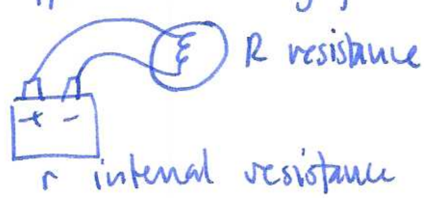
Thm Quotient rule (assume f, g differentiable, $g(x) \neq 0$)

$$\left(\frac{f}{g}\right)'(x) = \frac{gf' - fg'}{g^2} = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

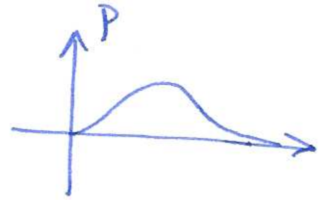
Examples ① $\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{(x+1)(x)' - x(x+1)'}{(x+1)^2} = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2}$

② $\frac{d}{dt}\left(\frac{e^t}{e^t+t}\right) = \frac{(e^t+t)(e^t)' - (e^t+t)'(e^t)}{(e^t+t)^2} = \frac{(e^t+t)e^t - (e^t+1)e^t}{(e^t+t)^2} = \frac{te^t - e^t}{(e^t+t)^2}$

③ application: battery power



power $P = \frac{V^2 R}{(r+R)^2}$



Q: when does the battery give maximal power?

A: when $\frac{dP}{dR} = 0$ $\quad P = \frac{V^2 R}{(r+R)}$ $\quad P(R)$, V, r constant

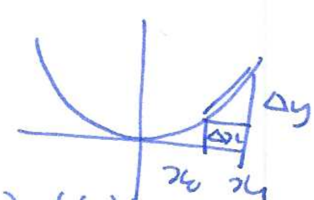
$$\frac{dP}{dR} = \frac{(r+R)^2(V^2R)' - ((r+R)^2)'(V^2R)}{(r+R)^4} = \frac{(r+R)^2V^2 - (r^2+2Rr+R^2)'V^2R}{(r+R)^4}$$

$$= \frac{V^2\left[(r+R)^2 - R(2r+2R)\right]}{(r+R)^4} = \frac{V^2(r+R)(r+R-2R)}{(r+R)^4} = \frac{V^2(r-R)}{(r+R)^3} = 0$$

$\Rightarrow R = r$. $\square$.

§3.4 Rates of change

recall: average rate of change $= \frac{\Delta y}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$



instantaneous rate of change: $f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_1 \to x} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$

observation: if Δx is small, we can use average rate of change to approximate actual rate of change, and vice versa.