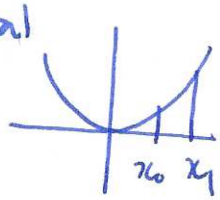
$$\left.\begin{array}{l} f(0) = +\infty > 0 \\ f(\frac{\pi}{2}) = \frac{2}{\pi} - 1 < 0 \end{array}\right\} \quad \text{midpoint } \frac{\pi}{4}, \quad f(\tfrac{\pi}{4}) = \frac{4}{\pi} - \sin(\tfrac{\pi}{4}) \approx 0.566 > 0$$

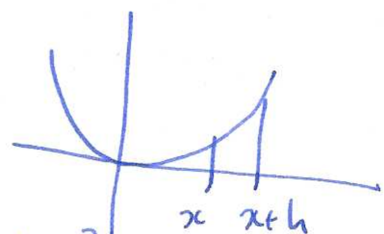
so continue with $[\frac{\pi}{4}, \frac{\pi}{2}]$, etc.

§3.1 Def^n of the derivative

Recall: we can compute the average rate of change of a function over an interval



$[x_0, x_1]$ $\quad \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$



$\dfrac{f(x+h) - f(x)}{h}$

Q: how do we find the slope of the tangent line?

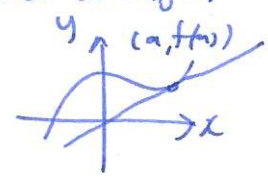
A: look at average rate of change over small interval $[x, x+h]$ and take limit as $h \to 0$.

Def^n the slope of the tangent line at $x = a$ is $\displaystyle\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

Notation: also called the derivative written $f'(a)$ (Newton) or $\dfrac{df}{dx}(a)$ (Leibniz)

If this limit exists, we say that the function $f(x)$ is differentiable at $x = a$. Note: $\displaystyle\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ same as $\displaystyle\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

Def^n the tangent line to $f(x)$ at the point $(a, f(a))$ is the straight line through $(a, f(a))$ with slope $f'(a)$



the equation for this line is:

$$y - y_0 = m(x - x_0)$$

$$y - f(a) = f'(a)(x - a)$$

or

$$y = f(a) + f'(a)(x - a)$$

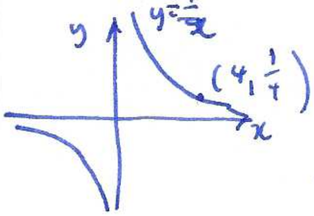
Example find the tangent line to $y = x^2$ at $x = 1$

$(x, f(x)) = (1, 1)$ $\quad$ slope $f'(1) = \displaystyle\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h)^2 - 1}{h}$

$$= \lim_{h\to 0} \frac{1+2h+h^2-1}{h} = \lim_{h\to 0} 2+h = 2$$

so equation of tangent line is $y-1 = 2(x-1)$ $\quad y = 1+2(x-1)$

<u>Example</u> find slope of tangent line to $f(x) = \frac{1}{x}$ at $x=4$



slope $f'(4) = \lim_{h\to 0} \frac{f(4+h)-f(4)}{h} = \lim_{h\to 0} \frac{\frac{1}{4+h} - \frac{1}{4}}{h}$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{4+h} - \frac{1}{4}\right) = \lim_{h\to 0} \frac{1}{h}\,\frac{4-(4+h)}{4(4+h)} = \lim_{h\to 0} \frac{-h}{h\,4(4+h)}$$

$$= \lim_{h\to 0} \frac{-1}{4(4+h)} = \frac{-1}{16}$$

tangent line: $y - \frac{1}{4} = -\frac{1}{16}(x-4)$

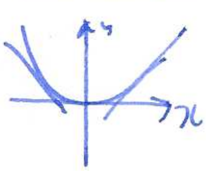
<u>Example</u> find slope of straight line $y = mx+b$

find slope at $x=a$: $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{m(a+h)+b-(ma+b)}{h}$

$$= \lim_{h\to 0} \frac{ma+mh+b-ma+b}{h} = \lim_{h\to 0} m = m.$$

<u>Observation</u> if $f(x) = b$ (constant) then $f'(x) = 0$ for all x.

<u>§2.2 Derivative as a function</u>



$f: \mathbb{R} \to \mathbb{R}$
$x \to x^2$

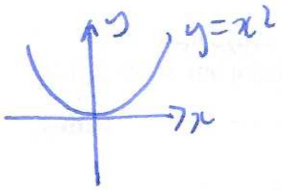
at each point x, there is a tangent line, with a slope, which is a number

Therefore we can define a function

$x \mapsto$ slope of tangent line at $(x, f(x))$

<u>notation</u>: we call this function $f'(x)$ or "the derivative of f"

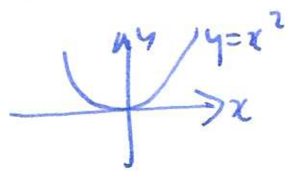
<u>Example</u>

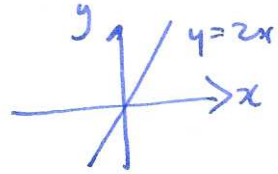


then $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ if $f(x) = x^2$

then $f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0} 2x+h = 2x$

<u>summary</u> if $f(x) = x^2$, then $f'(x) = 2x$

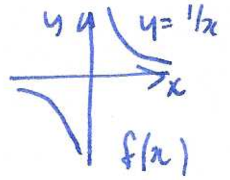


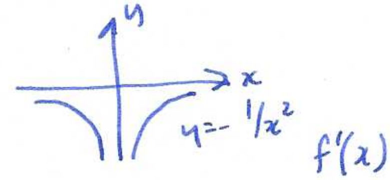


Example $f(x) = \frac{1}{x}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{1}{h}\,\frac{x-(x+h)}{(x+h)x} = \lim_{h\to 0} \frac{-h}{h(x+h)x}$$

$$= \lim_{x\to 0} \frac{-1}{(x+h)x} = -\frac{1}{x^2}$$





Remarks ① functions $\quad f: \text{domain} \to \text{range}$, e.g. $f: \mathbb{R} \to \mathbb{R}$

derivative : (differentiable) functions $\to$ functions

$$f(x) \longmapsto f'(x)$$

warning: not all functions differentiable

② "calculus" means rules for doing calculations. We don't have to explicitly compute limits all the time.

Example $f(x) = x^3 \quad f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$

$$= \lim_{h\to 0} 3x^2 + 3xh + h^2 = 3x^2$$

Thm (powers of x) if $f(x) = x^n$, $f'(x) = nx^{n-1}$ (works for all $n \in \mathbb{R}$!)

Examples $\frac{d}{dx}(x^2) = 2x \quad \frac{d}{dx}(x^3) = 3x^2 \quad \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -x^{-2}$

$$\frac{d}{dx}(x^1) = 1 \quad \frac{d}{dx}(1 = x^0) = 0 \quad \frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

Thm $\frac{d}{dx}(x^n) = nx^{n-1}$

Proof $f(x) = x^n \quad f'(x) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$ $\qquad \binom{n}{k} = \frac{n!}{k!(n-k)!}$

binomial theorem : $(x+h)^n = x^n + nx^{n-1}h + \underbrace{\binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-1}xh^{n-1} + h^n}_{\text{all of these contain } h^2}$

$$\text{so } \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h\to 0} \frac{\cancel{x^n} + nx^{n-1}h + O(h^2) - \cancel{x^n}}{h} = \lim_{h\to 0} nx^{n-1}\cancel{h} + h(\ldots) = nx^{n-1}$$

$\square$