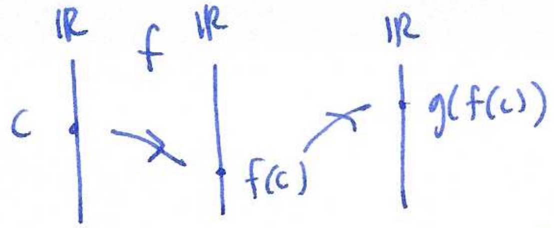


Example $f(x) = \dfrac{2^x + \sin(x)}{\sqrt{x^2+x+1}}$ cts at $x=1$

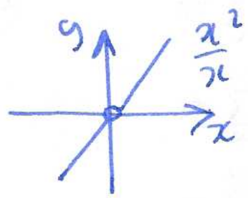
Q: where is $f(x) = \dfrac{x^2}{\sin(x)}$ cts ?

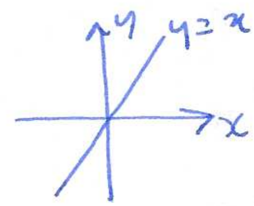
§2.5 Evaluating limits algebraically

Example $\dfrac{x^2}{x}$ undefined at $x=0$, $\dfrac{0}{0}$ indeterminate form

but $\lim\limits_{x\to 0} \dfrac{x^2}{x}$ does not depend on value at $x=0$

$\dfrac{x^2}{x} = x$ for $x \neq 0$





so $\lim\limits_{x\to 0} \dfrac{x^2}{x} = \lim\limits_{x\to 0} x = 0$

indeterminate forms: $\dfrac{0}{0}, \dfrac{\infty}{\infty}, \infty\cdot 0, \infty-\infty, 0^0$

note $\dfrac{1}{0}$ not indeterminate, limit will be $\pm\infty$ or DNE.

Examples ① $\lim\limits_{x\to 3} \dfrac{x^2-4x+3}{x^2+x-12}$ $\quad x=3: \quad \dfrac{9-12+3}{9+3-12} = \dfrac{0}{0}$

factor : $\dfrac{(x-3)(x-1)}{(x-3)(x+4)} = \dfrac{x-1}{x+4} \quad (x\neq 3)$

$$\lim_{x\to 3} \frac{x^2-4x+3}{x^2+x-12} = \lim_{x\to 3} \frac{x-1}{x+4} = \frac{2}{7}$$

② $\lim\limits_{x\to 4} \dfrac{\sqrt{x}-2}{x-4}$ $\quad x=4: \quad \dfrac{2-2}{4-4} = \dfrac{0}{0}$

$$\frac{\sqrt{x}-2}{x-4} = \frac{\sqrt{x}-2}{(\sqrt{x}-2)(\sqrt{x}+2)} = \frac{1}{\sqrt{x}+2} \quad (x\neq 4)$$

$$\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{4}$$

③ $\lim_{x\to\frac{\pi}{2}} \dfrac{\tan x}{\sec x} = \lim_{x\to\frac{\pi}{2}} \dfrac{\sin x/\cos x}{1/\cos x} = \lim_{x\to\frac{\pi}{2}} \sin x = 1$

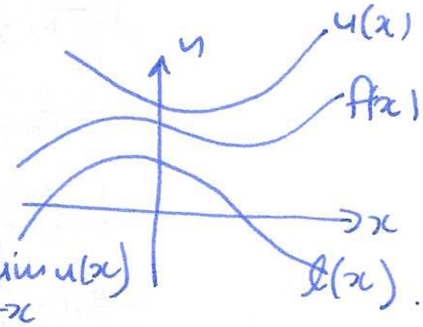
④ $\lim_{x\to 1} \dfrac{1}{x-1} - \dfrac{2}{x^2-1} = \dfrac{x+1-2}{x^2-1} = \dfrac{x-1}{x^2-1} = \dfrac{x-1}{(x+1)(x-1)} = \dfrac{1}{x+1} \quad x\neq 1$

$= \lim_{x\to 1} \dfrac{1}{x+1} = \dfrac{1}{2}$

§2.6 Trigonometric limits

Q: what is $\lim_{x\to 0} \dfrac{\sin x}{x}$? substitute $x=0$, get $\frac{0}{0}$ indeterminate form.

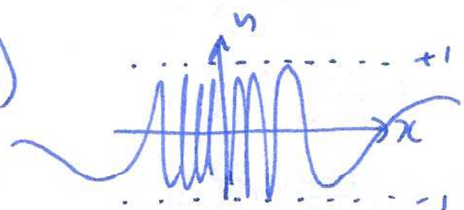
A: $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$ (try : $x = 0.1, 0.01, \ldots$)



squeeze theorem : suppose $l(x) \le f(x) \le u(x)$

Thm suppose $l(x) \le f(x) \le u(x)$ and $\lim_{x\to c} l(x) = L = \lim_{x\to c} u(x)$ then $\lim_{x\to c} f(x) = L$

Example $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$



$y = \sin\left(\frac{1}{x}\right)$

$y = x\sin\left(\frac{1}{x}\right)$

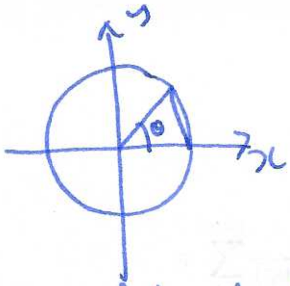
$-1 \le \sin(x) \le 1$

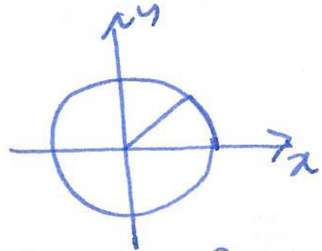
$-|x| \le x\sin(x) \le |x|$

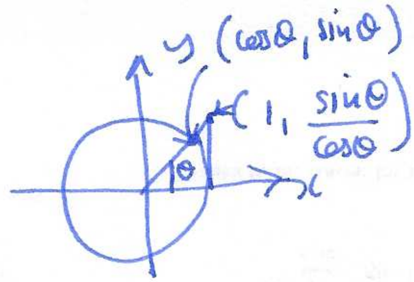
$\lim_{x\to 0} |x| = 0 \quad \lim_{x\to 0} -|x| = 0 \rightarrow \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$

Thm $\lim_{x\to 0} \dfrac{\sin x}{x} = 1 \qquad \lim_{x\to 0} \dfrac{1-\cos x}{x} = 0$

Proof (of $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$) assume $0 < \theta < \frac{\pi}{2}$. consider the following three areas :







area of triangle $\frac{1}{2}bh$ $\leq$ area of sector $\leq$ area of triangl $\frac{1}{2}bh$

$\frac{1}{2} \cdot 1 \cdot \sin\theta$ $\quad$ $\cancel{\pi} \frac{\theta}{2\cancel{\pi}}$ $\quad$ $\frac{1}{2} \frac{\sin\theta}{\cos\theta}$

$$\frac{1}{2}\sin\theta \leq \frac{1}{2}\theta \leq \frac{1}{2}\frac{\sin\theta}{\cos\theta}$$

$$\frac{\sin\theta}{\theta} \leq 1 \qquad \cos\theta \leq \frac{\sin\theta}{\theta}$$

so

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1 \qquad \lim_{\theta\to 0} 1 = 1 \qquad \lim_{\theta \to 0}\cos\theta = 1$$

squeeze thm $\Rightarrow \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ $\square$

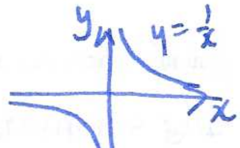
Examples ① $\lim_{x\to 0} \frac{\sin 3x}{2x}$ $\qquad$ know: $\lim_{x\to 0} \frac{\sin x}{x} = 1$

write $3x = \theta$ $\quad \lim_{x\to 0} \frac{\sin\theta}{2 \cdot x/3} = \lim_{x\to 0} \frac{3}{2}\frac{\sin\theta}{\theta} = \frac{3}{2}\lim_{x\to 0}\frac{\sin\theta}{\theta} = \frac{3}{2}$

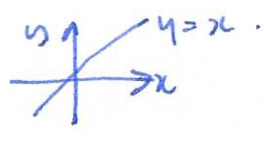
② $\lim_{t\to 0} \frac{1-\cos t}{\sin t} = \lim_{t\to 0} \frac{1-\cos t}{t} \cdot \frac{t}{\sin t} = \underbrace{\lim_{t\to 0}\frac{1-\cos t}{t}}_{0} \frac{1}{\underbrace{\lim_{t\to 0}\frac{\sin t}{t}}_{=1}} = 0$

§2.7 Limits at infinity

key observation: $\lim_{x\to\infty} \frac{1}{x} = 0$



$\lim_{x\to\infty} x = \infty$



Examples ① $\lim\limits_{x\to\infty} \frac{3x}{2x-1} = \lim\limits_{x\to\infty} \frac{3}{2-1/x} = \frac{3}{2}$

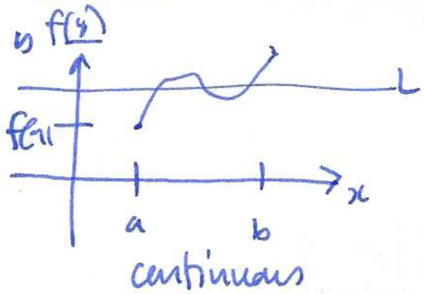
② $\lim\limits_{x\to\infty} \frac{x^2+x}{x-3} = \lim\limits_{x\to\infty} \frac{x+1}{1-3/x} = +\infty$

③ $\lim\limits_{x\to\infty} \frac{1}{x} - \frac{2}{3x+1} = \lim\limits_{x\to\infty} \frac{1}{x} - \lim\limits_{x\to\infty} \frac{2}{3x+1} = 0 - 0 = 0$

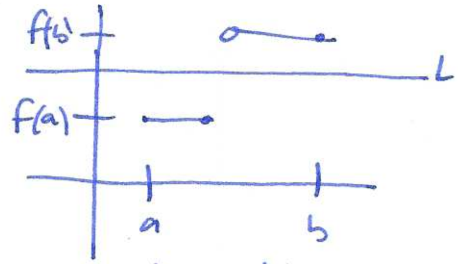
④ $\lim\limits_{x\to\infty} \frac{\sqrt{2x^2+1}}{4x+1} = \lim\limits_{x\to\infty} \frac{\sqrt{2+1/x^2}}{4+1/x} = \frac{\sqrt{2}}{4}$ (do $\lim\limits_{x\to-\infty}$).

§2.8 Intermediate Value Theorem (IVT)

"continuous functions can't skip values"



continuous

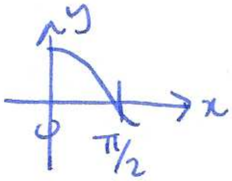


not continuous

Thm (Intermediate value theorem IVT).

If $f(x)$ is a cts function on a closed interval $[a,b]$ with $f(a) \neq f(b)$ then for any number L between $f(a)$ and $f(b)$ there is at least one $c \in [a,b]$ s.t. $f(c) = L$.

Example show $\cos(x) = \frac{1}{4}$ has at least one solution



consider $(0, \frac{\pi}{4})$ $\cos(0) = 1$ $\cos(\frac{\pi}{4}) = 1$ $0 \leq \frac{1}{4} \leq 1$

$\Rightarrow \exists\, c$ s.t. $\cos(c) = \frac{1}{4}$.

special case : finding zeros

Corollary if $f(x)$ is cts on $[a,b]$, and $f(a), f(b)$ have different signs, then there is at least one $c \in [a,b]$ with $f(c) = 0$.

Bisection method : find a solution to $\sin(x) = \frac{1}{x}$ in $[0, \frac{\pi}{2}]$

consider $f(x) = \frac{1}{x} - \sin x$