calculate:

x	$\frac{x-9}{\sqrt{x}-3}$
8.9	5.983
9.1	6.016
8.99	5.998
9.01	6.002

algebra: difference of two squares

$$x-9 = (\sqrt{x})^2 - 3^2 = (\sqrt{x}-3)(\sqrt{x}+3)$$

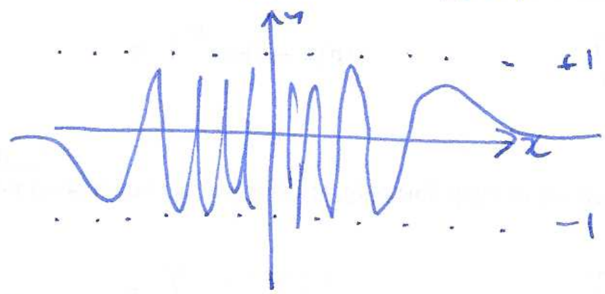
$$\frac{x-9}{\sqrt{x}-3} = \frac{(\sqrt{x}-3)(\sqrt{x}+3)}{\sqrt{x}-3} = \sqrt{x}+3 \quad (x \neq 9)$$

$$\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3} = \lim_{x\to 9} \sqrt{x}+3 = 6$$

Bad example: no limit

$f(x) = \sin(\frac{1}{x})$

no limit at $x=0$

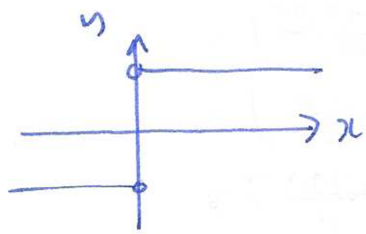


note: $f\left(\frac{1}{2\pi n}\right) = \sin(2\pi n) = 0$

$f\left(\frac{1}{2\pi n + \frac{\pi}{2}}\right) = \sin\left(2\pi n + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$

One sided limits

example $f(x) = \frac{x}{|x|}$ $\qquad f(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases}$



sometimes useful to distinguish left limit / right limit / two sided limit

notation $\lim_{x\to 0+} f(x)$ means right limit (only consider $x > 0$)

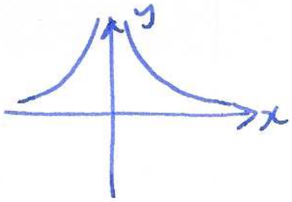
$\lim_{x\to 0-} f(x)$ means left limit (only consider $x < 0$)

note: in order for the two sided limit $\lim_{x\to 0} f(x)$ to exist, the left and right limits must exist and be equal.

example $f(x) = \frac{x}{|x|}$

$$\left.\begin{array}{l} \lim_{x\to 0+} f(x) = +1 \\ \lim_{x\to 0-} f(x) = -1 \end{array}\right\} 1 \neq -1 \text{ so } \lim_{x\to 0} f(x) \text{ DNE}$$

Example $f(x) = \frac{1}{x^2}$



$$\left.\begin{array}{l} \lim_{x\to 0^+} \frac{1}{x^2} = +\infty \\ \lim_{x\to 0^-} \frac{1}{x^2} = +\infty \end{array}\right\} \lim_{x\to 0} \frac{1}{x^2} = +\infty$$

§2.3 Basic limit laws

Example $\lim_{x\to 0} 2x+2 = \lim_{x\to 0} 2x + \lim_{x\to 0} 2 = 2\lim_{x\to 0} x + \lim_{x\to 0} 2 = 2$

Thm assume that $\lim_{x\to c} f(x)$, $\lim_{x\to c} g(x)$ exist and are finite. Then:

1) sums: $\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$

2) constant multiple: $\lim_{x\to c} k f(x) = k \lim_{x\to c} f(x)$ $\quad$ k constant (does not depend on x)

3) products: $\lim_{x\to c}(f(x)g(x)) = (\lim_{x\to c} f(x))(\lim_{x\to c} g(x))$

4) quotients: $\lim_{x\to c} \frac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ $\quad$ as long as $\lim_{x\to c} g(x) \neq 0$.

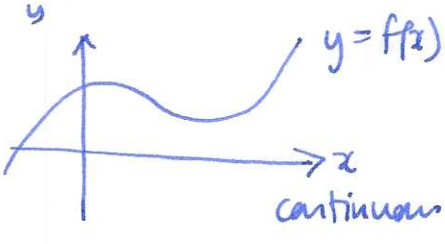
warning: these rules don't work if either $\lim_{x\to c} g(x)$, $\lim_{x\to c} f(x)$ DNE.

Examples: $\lim_{x\to 3} x^2 = \lim_{x\to 3} x \cdot \lim_{x\to 3} x = 3.3 = 9$.

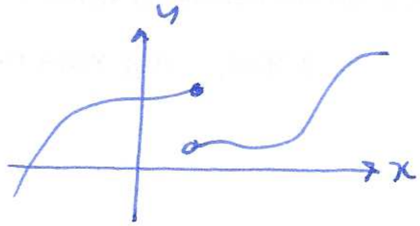
$$\lim_{t\to 2} \frac{t+5}{2t} = \frac{\lim_{t\to 2} t+5}{\lim_{t\to 2} 2t} = \frac{7}{4}$$

§2.4 Limits and continuity

Example



continuous

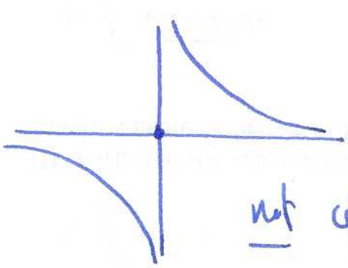


not continuous / discontinuous

<u>Def^n</u> we say $f(x)$ is continuous at $x=c$ if $\lim_{x\to c} f(x) = f(c)$

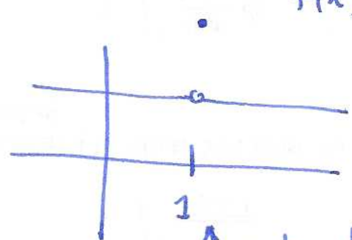
If the limit DNE, is infinite, is not equal to $f(c)$, then $f(x)$ is <u>not</u> continuous at $x=c$.

<u>Examples</u>



$f(x) = \begin{cases} 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$

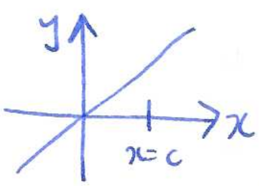
<u>not</u> cts at $x=0$



$f(x) = \begin{cases} 1 & x \neq 1 \\ 2 & x = 1 \end{cases}$

1 ↑ not cts at $x=1$

<u>Example</u> show $f(x) = x$ is cts.



$f(c) = c$, want to show $\lim_{x\to c} f(x) = c$

follows from limit law: $\lim_{x\to c} x = c$ ✓

<u>Corollary</u> polynomials / rational functions are cts where defined.

<u>Def^n</u> $f(x)$ is <u>left continuous</u> at $x=c$ if $\lim_{x\to c^-} f(x) = f(c)$

<u>right continuous</u> at $x=c$ $\lim_{x\to c^+} f(x) = f(c)$

If at least one of the left or right limits is $\pm\infty$ we say $f(x)$ has an infinite discontinuity at $x=c$.

<u>Building continuous functions</u>

<u>Thm</u> 0 $f(x) = k$, $f(x) = x$ are continuous

<u>Thm</u> 1 suppose $f(x), g(x)$ both continuous at $x=c$, then the following functions are cts at $x=c$:

1) $f(x) + g(x)$ 2) $kf(x)$ for any constant k

3) $f(x)g(x)$ 4) $\dfrac{f(x)}{g(x)}$ if $g(c) \neq 0$

<u>Proof</u>: these follow directly from the limit laws

check 1) $f(x)$ cts means $\lim_{x \to c} f(x) = f(c)$

$g(x)$ cts means $\lim_{x \to c} g(x) = g(c)$

so $\lim_{x \to c} \left(f(x) + g(x) \right) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = f(c) + g(c)$
as required $\square$.

<u>Thm 2</u> Polynomials are continuous, $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$
Rational functions $\frac{p(x)}{q(x)}$ are cts, except where $q(x) = 0$.

<u>Proof</u> $f(x) = x$ is cts. So $f(x) . f(x) = x . x = x^2$ is cts. (products)
similarly, x^n is cts.
so $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ is cts (multiplication by a constant and addition)
so $\frac{p(x)}{q(x)}$ is cts (quotients) where $q(x) \neq 0$ $\square$.

<u>useful facts</u>

<u>Thm 3</u>
- $\sin(x)$, $\cos(x)$ are continuous
- b^x is cts
- $\log_b(x)$ is cts
- $x^{1/n}$ is cts

(combinations of these with polynomials are sometimes called elementary functions)

<u>Thm 4</u> (inverse functions) If $f : D \to \mathbb{R}$ is continuous, with inverse $f^{-1} : \mathbb{R} \to D$, then f^{-1} is continuous.

<u>Thm 5</u> (composition) If $f(x)$ is cts at $x = c$, and $g(x)$ is cts at $x = f(c)$, then $g(f(x))$ is cts at $x = c$