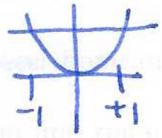
§1.5 Inverse functions

recall : $f : \mathbb{R} \to \mathbb{R}$
domain range
$x \mapsto f(x)$

want: the inverse function should be the reverse of this
$\mathbb{R} \xleftarrow{f^{-1}} \mathbb{R}$
$f^{-1}(x) \leftarrow x$
$x \leftarrow f(x)$

<u>problem</u> : the inverse is often not a function

<u>example</u> $f : \mathbb{R} \to \mathbb{R}$
$x \mapsto x^2$



$f(+1) = 1$
$f(-1) = 1$

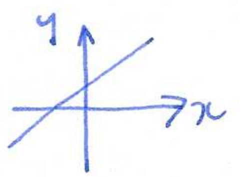
Q: what is $f^{-1}(1)$?

<u>Q</u>: when does a function have an inverse ?

<u>A</u>: when it passes the <u>horizontal line test</u> (one-to-one / injective)

$\Leftrightarrow$ for each number $c \in$ range, there is a unique x s.t. $f(x) = c$

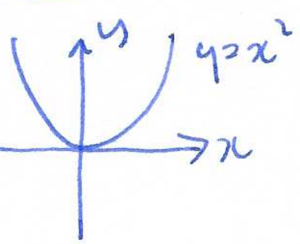
<u>Example</u> $y = x + 1$



Q: how do we find a formula for the inverse.

<u>A</u>: ① write down $y = f(x)$
② solve for x in terms of y, i.e. $x = g(y)$
③ $f^{-1}(x) = g(x)$
④ check!

<u>Example</u> $f(x) = x^2$



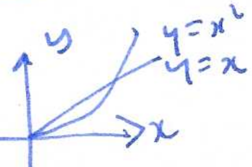
<u>problem</u>: doesn't pass horizontal line test

<u>fix</u>: restrict domain, consider $f : [0, \infty) \to [0, \infty)$
$x \mapsto x^2$

<u>does</u> pass horizontal line test
so has an inverse we call $\sqrt{x}$

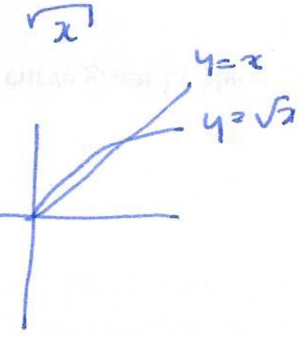
$f^{-1} : [0, \infty) \mapsto [0, \infty)$
$x \mapsto \sqrt{x}$

<u>How to draw the graph of the inverse</u>



• reflect in $y = x$

reason: graph of f is pairs $(x, f(x))$
f^{-1} $(x, f^{-1}(x))$

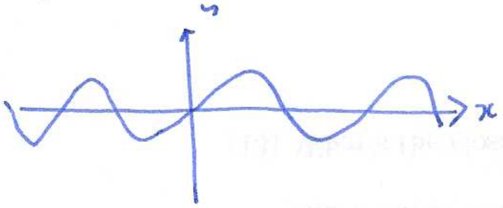


$y = f(x) \leftrightarrow f^{-1}(y) = x$ $\quad (x, f(x)) \leftrightarrow (f^{-1}(y), y)$

$\leftrightarrow$ swap and relabel.

Inverse trig functions

$y = \sin(x)$

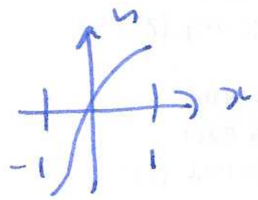


problem: not one-to-one

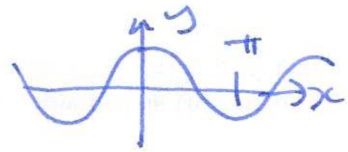
fix: restrict domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$\sin(x) : [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow [-1, 1]$

$\sin^{-1}(x) : [-1, 1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$



similarly $y = \cos(x)$



restrict domain to $[0, \pi]$

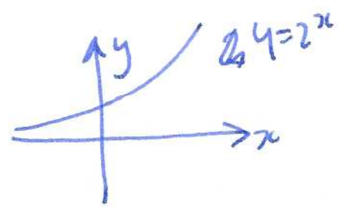
$\cos(x) : [0, \pi] \longrightarrow [-1, 1]$

$\cos^{-1}(x) : [-1, 1] \longrightarrow [0, \pi]$

§1.6 Exponential and logarithm functions

Example $x \mapsto 2^x$

x	-2	-1	0	1	2	3
2^x	$1/4$	$1/2$	1	2	4	8



can use any positive number instead of 2, $f(x) = b^x$, $b > 0$

useful properties:
- positive, $b > 0$
- b^x increasing if $b > 1$
  decreasing if $b < 1$
- b^x grows faster than any polynomial

exponent rules $\quad b^0 = 1 \quad b^x b^y = b^{x+y} \quad b^{-x} = \frac{1}{b^x} \quad \frac{b^x}{b^y} = b^{x-y}$

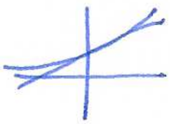
$(b^x)^y = b^{xy} \quad b^{1/n} = \sqrt[n]{b}$

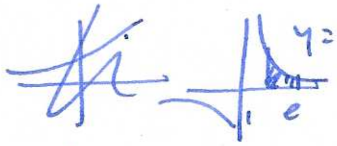
- there is a special exponential function e^x, $e = 2.71828\ldots$

key properties of e^x

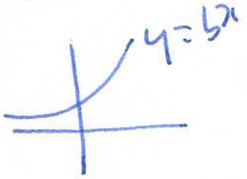
① e is the unique number s.t. e^x has slope 1 at $x=0$

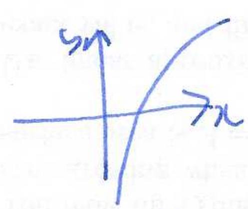
② e is the unique number s.t. the area under the curve $y=\frac{1}{x}$ between 1 and e has area 1.





logarithms the logarithm is the inverse function for the exponential function





the special logarithm with base $b=e$ is called the natural logarithm $\ln(x)$

• inverse function properties $f^{-1}(f(x)) = x = f(f^{-1}(x))$

$$b^{\log_b x} = x = \log_b(b^x)$$

• logarithm rules : $\log_b(1) = 0 \quad \log_b(b) = 1$

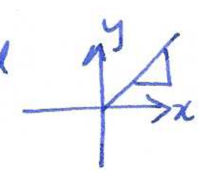
$\log_b(st) = \log_b(s) + \log_b(t) \qquad \log_b\left(\frac{1}{t}\right) = -\log_b(t)$

$\log_b\left(\frac{s}{t}\right) = \log_b(s) - \log_b(t) \qquad \log_b(s^t) = t\log_b(s)$

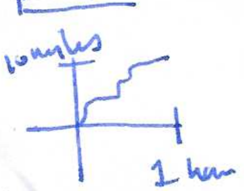
base conversion : $\log_b(x) = \dfrac{\log_a(x)}{\log_a(b)}$ for any a, so $\log_b(x) = \dfrac{\ln(x)}{\ln(b)}$

§2.1 Limits, rates of change, tangent lines

motivation : velocity $= \dfrac{\text{distance}}{\text{time}}$ example: driving at constant speed velocity = slope of line.



problem : what happens if you don't drive at constant speed?



average speed $= \dfrac{\text{distance travelled}}{\text{time}}$

we can look at average speed over any time interval, including very short ones.