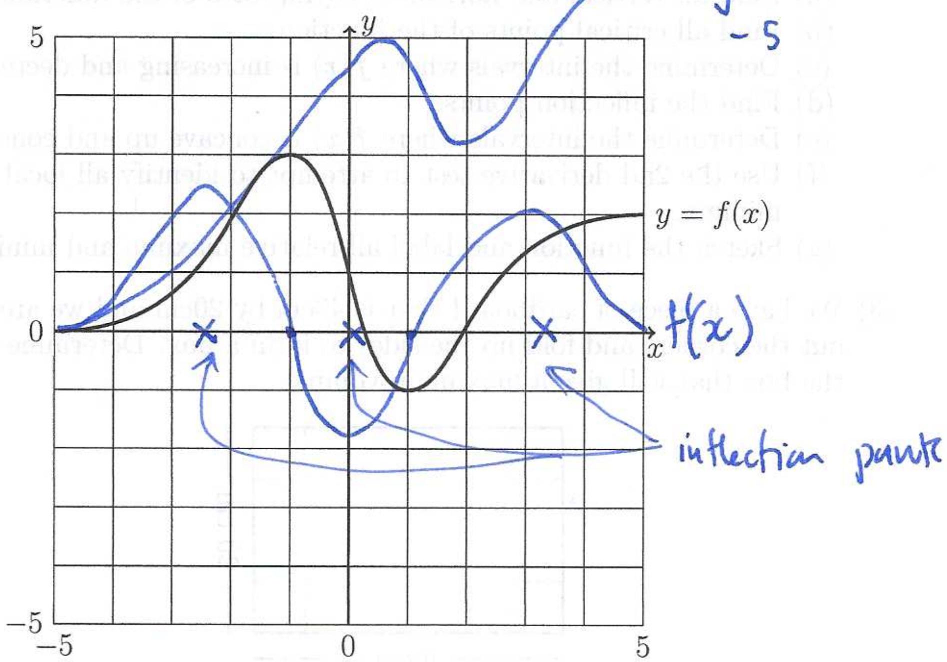
Math 231 Calculus 1 Spring 22 Sample Midterm 3

(1) Consider the function $f(x)$ defined by the following graph. $\int_{-5}^{x} f(t)\,dt$



(a) Label all regions where $f'(x) < 0$. $(-1, 1)$

(b) Label all regions where $f'(x) > 0$. $(-5, -1) \cup (1, 5)$

(c) What is $\lim_{x\to\infty} f(x)$? 2

(d) What is $\lim_{x\to-\infty} f'(x)$? 0

(e) What is $\lim_{x\to\infty} f''(x)$? 0

(f) Sketch a graph of $f'(x)$ on the figure.

(g) Sketch a graph of $\int_{-5}^{x} f(t)dt$ on the figure.

(h) Label the approximate locations of all points of inflection. X

SMT3 Solutions

Q2 $f(x) = e^{9-x^2}$ a) vertical asymptotes: none

horizontal asymptotes $\lim_{x \to \pm\infty} 9-x^2 = -\infty$ so $\lim_{x\to\pm\infty} e^{9-x^2} = 0$

two horizontal asymptotes, both 0.

b) $f'(x) = e^{9-x^2} \cdot (-2x)$ $\quad f'(x) = 0 \Rightarrow x = 0$ $\quad$ 1 critical pt at $x=0$

c) $e^{9-x^2} > 0$ so $f'(x) > 0$ when $x < 0$

$f'(x) < 0$ $\quad x > 0$

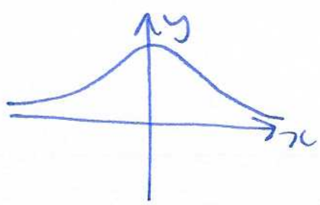
d) $f''(x) = e^{9-x^2}(-2x)^2 + e^{9-x^2}(-2) = 2e^{9-x^2}(2x^2-1)$ $\quad x = \pm\frac{1}{\sqrt{2}}$.

e) $\begin{array}{ccccc} + & & - & & + \\ & -1/\sqrt{2} & & 1/\sqrt{2} & \end{array}$ $f''(x)$ $\quad$ concave up $(-\infty, -\frac{1}{\sqrt{2}}) \cup (\frac{1}{\sqrt{2}}, \infty)$

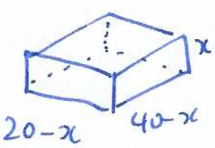
down $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

f) $f''(0) = -2e^9 < 0 \Rightarrow$ local max

g)



Q3

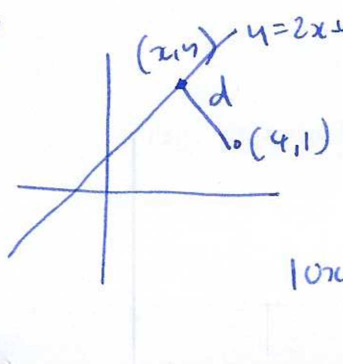


$V = x(20-x)(40-x) = 800x - 60x^2 + x^3$

$\frac{dV}{dx} = 800 - 120x + 3x^2$ $\quad x = \frac{120 \pm \sqrt{120^2 - 4\cdot 3\cdot 800}}{2\cdot 3}$.

$x = 20 \pm \frac{2\sqrt{10}}{3}$. $= 20 - \frac{2\sqrt{10}}{3} \approx$ ~~[illegible]~~. 8.453

Q4



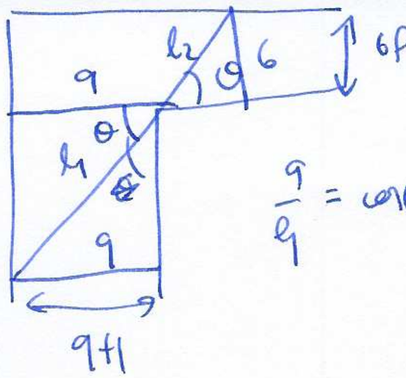
$d^2 = (4-x)^2 + (1-y)^2 = (4-x^2) + (1-2x+1)^2$

$= (4-x)^2 + (2-2x)^2$

$\frac{d(d^2)}{dx} = 2(4-x)(-1) + 2(2-2x)(-2) = -8 + 2x - 8 + 8x = 0$

$10x = 16$ $\quad x = 1.6, y = 4.2$.

Q5



$$l = l_1 + l_2 = \frac{9}{\cos\theta} + \frac{6}{\sin\theta}$$

$$\frac{6}{l_2} = \sin\theta \qquad \frac{9}{l_1} = \cos\theta$$

$$\frac{dl}{d\theta} = -9(\cos\theta)^{-2}\cdot(-\sin\theta) + -6(\sin\theta)^{-2}\cdot\cos\theta = 0$$

$$\frac{9\sin\theta}{\cos^2\theta} - \frac{6\cos\theta}{\sin^2\theta} = 0 \Rightarrow \frac{\sin^3\theta}{\cos^3\theta} = \frac{6}{9} = \frac{2}{3}.$$

$\theta = \tan^{-1}\left(\sqrt[3]{2/3}\right)$ $\quad \theta \approx 0.718 \quad l \approx 21.070$

Q6

a) $\overset{l'H}{=} \lim_{x\to 3} \frac{6x-7}{8x-13} = \frac{11}{11} = 1$

b) $\overset{l'H}{=} \lim_{x\to 0} \frac{4\sec^2 4x}{3} = \frac{4}{3}.$

c) $\overset{l'H}{=} \lim_{x\to 4} \frac{-\frac{1}{2}x^{-1/2}}{1} = -\frac{1}{4}.$

d) $\overset{l'H}{=} \lim_{x\to 6} \frac{4x-9}{\frac{1}{2}(x+3)^{-1/2}\cdot 1} = \frac{15}{1/6} = 90$

e) $\overset{l'H}{=} \lim_{x\to 3} \frac{-4x}{-x^{-2}} = \frac{+12}{1/9} = 108$

f) $\overset{l'H}{=} \lim_{x\to 0} \frac{\frac{1}{1+(3x)^2}\cdot 3}{\frac{1}{\sqrt{1-(5x)^2}}\cdot 5} = \frac{3}{5}.$

g) $\overset{l'H}{=} \lim_{x\to 0} \frac{2x e^{x/3} + x^2 e^{x/3}\cdot 1/3}{2\tan\left(\frac{2x}{5}\right)\cdot\sec^2\left(\frac{2x}{5}\right)\cdot\frac{2}{5}}$

$\overset{l'H}{=} \lim_{x\to 0} \frac{2e^{x/3} + 2xe^{x/3}\cdot 1/3 + 2xe^{x/3}\cdot 1/3 + x^2e^{x/3}\cdot 1/9}{\frac{4}{5}\left(\sec^2\left(\frac{2x}{5}\right)\cdot\frac{2}{5}\sec^2\left(\frac{2x}{5}\right) + \tan\left(\frac{2x}{5}\right)\cdot 2\sec\left(\frac{2x}{5}\right)\cdot\sec\left(\frac{2x}{5}\right)\tan\left(\frac{2x}{5}\right)\cdot\frac{2}{5}\right)}$

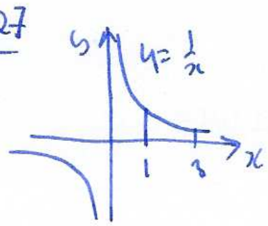
$= \frac{2}{4/5} = \frac{10}{8} = \frac{5}{4}.$

h) $= \lim_{x\to\infty} \frac{2 - 5/x + 7/x^2 - 2/x^3}{3 + 4/x - 21/x^3} = \frac{2}{3}$

i) $\overset{l'H}{=} \lim_{x\to\infty} \frac{e^{x^2}\cdot 2x}{2x+1} = \lim_{x\to\infty} \frac{2e^{x^2}}{2+\underbrace{(1/x)}_{\to 0}} = +\infty.$

j) $\lim_{x\to\frac{1}{2}^-} \frac{\tan(\pi x)}{\ln(1-2x)} \overset{l'H}{=} \lim_{x\to\frac{1}{2}^-} \frac{\sec^2(\pi x)\cdot\pi}{\frac{1}{1-2x}(-2)} = \lim_{x\to\frac{1}{2}} \frac{(4x-2)\pi}{\cos^2(\pi x)} \overset{l'H}{=} \lim_{x\to\frac{1}{2}^-} \frac{4\pi}{2\cos(\pi x)\sin(\pi x)\cdot\pi} = +\infty.$

Q7



$$L_4 = \frac{1}{2}\left(f(1) + f(1.5) + f(2) + f(2.5)\right).$$

$$= \frac{1}{2}\left(1 + \frac{2}{3} + \frac{1}{2} + \frac{2}{5}\right) = \frac{77}{60}$$

Q8 a) $\int 2x^{-1/3} - 3x^{2/3} + x^{5/3}\, dx = 3x^{2/3} - \frac{9}{5}x^{5/3} + \frac{3x^{8/3}}{8} + c$

b) $\int_{-1}^{3} |x| dx = \int_{-1}^{0} -x\, dx + \int_{0}^{3} x\, dx = \left[-\frac{1}{2}x^2\right]_{-1}^{0} + \left[\frac{1}{2}x^2\right]_{0}^{3} = \frac{1}{2} + \frac{9}{2} = 5$

c) $\int_{1}^{8} 3x^{-1/3} dx = \left[\frac{9x^{2/3}}{2}\right]_{1}^{8} = \frac{9}{2}(4-1) = \frac{27}{2}$

d) $\int_{0}^{4} e^{-3x} dx = \left[-\frac{1}{3}e^{-3x}\right]_{0}^{4} = -\frac{1}{3}e^{-12} + \frac{1}{3}$.

e) $\int_{0}^{x} \frac{1}{t+3} = \left[\ln(t+3)\right]_{0}^{x} = \ln(x+3) - \ln(3)$.

f) $\int \frac{1}{4+x^2} dx = \frac{1}{4}\int \frac{1}{1+(x/2)^2} dx$ $\quad u = \frac{x}{2}$, $\frac{du}{dx} = \frac{1}{2}$ $\quad \frac{1}{4}\int \frac{1}{1+u^2}\frac{dx}{du} du = \frac{1}{2}\tan^{-1}(u) + c$

$= \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

g) $\int \frac{x}{1+2x^2} dx$ $\quad u = 1+2x^2$, $\frac{du}{dx} = 4x$ $\quad \int \frac{x}{u}\cdot\frac{dx}{du} du = \int \frac{x}{u}\frac{1}{4x} du = \frac{1}{4}\ln(u) + c$

$= \frac{1}{4}\ln(1+2x^2) + c$

h) $\int \cos(5x) dx = \frac{1}{5}\sin(5x) + c$

i) $\int x \sin(1+x^2) dx$ $\quad u = 1+x^2$, $\frac{du}{dx} = 2x$ $\quad \int x \sin(u)\frac{dx}{du} du = \int x \sin(u)\cdot\frac{1}{2x} du$

$= \frac{1}{2}\int \sin(u) du = -\frac{1}{2}\cos(u) + c = -\frac{1}{2}\cos(1+x^2) + c$

j) $\int \frac{\cos(x)}{e^{\sin(x)}} dx$ $\quad u = \sin x$, $\frac{du}{dx} = \cos x$ $\quad \int \cos x \cdot e^{-u}\frac{dx}{du} du = \int \cos x \cdot e^{-u}\frac{1}{\cos x} du$

$$= \int e^{-u}\,du = -e^{-u}+c = -e^{-\sin x}+c$$

k) $\int \frac{\sin(x)}{\cos^2(x)}\,dx$ $\quad u=\cos x \quad \frac{du}{dx}=-\sin x$ $\quad \int \frac{\sin x}{u^2}\cdot\frac{1}{-\sin x}\,du = \int -u^{-2}\,du$

$$u^{-1}+c = \frac{1}{\cos x}+c = \sec(x)+c$$

Q9 $\quad v(t) = (t+1)^{-4}$

$x(t) = -\frac{1}{3}(t+1)^{-3}+c \qquad x(0)=0 = -\frac{1}{3}+c=0 \Rightarrow c=\frac{1}{3}.$

$x(t) = \frac{1}{3}\left(1-(t+1)^{-3}\right)$ as $t\to\infty$ $\quad x(t)\to\frac{1}{3}$, does not reach $x=10$.