SMT1 Solutions MTH 231 Spring 2022

Q1 a) 1 b) $\mathbb{R}$ c) DNE d) -3 e) -3 f) 1

Q2 a) $\lim\limits_{x\to -3} \frac{(x+3)(x-2)}{x+3} = \lim\limits_{x\to -3} x-2 = -5$

b) $\lim\limits_{x\to\infty} \frac{\sqrt[3]{4+3/x^2}}{3/x - 2} = \frac{\sqrt[3]{4}}{-2}$ ($3/x^2 \to 0$, $3/x \to 0$)

c) $\lim\limits_{x\to 0} \frac{\sin 3x}{4x}$ $\quad 3x=\theta \quad$ $\lim\limits_{\theta\to 0} \frac{\sin\theta}{4\theta/3} = \frac{3}{4}$

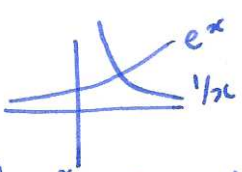
d) $\lim\limits_{x\to 0} \frac{x}{3-\sqrt{x+9}} \cdot \frac{3+\sqrt{x+9}}{3+\sqrt{x+9}} = \lim\limits_{x\to 0} \frac{3x + x\sqrt{x+9}}{9-x-9} = \lim\limits_{x\to 0} -3 - \sqrt{x+9} = -6$

Q3 $\lim\limits_{x\to 2} 3x - \frac{1}{x+2} = 6 - \frac{1}{4} = 5\frac{3}{4}$ $\qquad \lim\limits_{x\to 2} \frac{\cos(\pi x)}{x} = \frac{1}{2}$

$\frac{1}{2} \neq 5\frac{3}{4}$ so no value of c makes f cts.

Q4 $\frac{V(5)-V(4)}{5-1} = \frac{4}{3}\pi(5^3-4^3) = \frac{4}{3}\pi(125-64) = \frac{4}{3}\pi \cdot 61$.

Q5



$\lim\limits_{x\to 0+} \frac{1}{x} - e^x = +\infty$, $e^2 > \frac{1}{2}$, so there is $x>0$, s.t. $f(x) = \frac{1}{x} - e^x > 0$ and $f(2) < 0$, so IVT $\Rightarrow \exists\, x \in (0,2)$ s.t. $e^x = \frac{1}{x}$.

Q6 see next page.

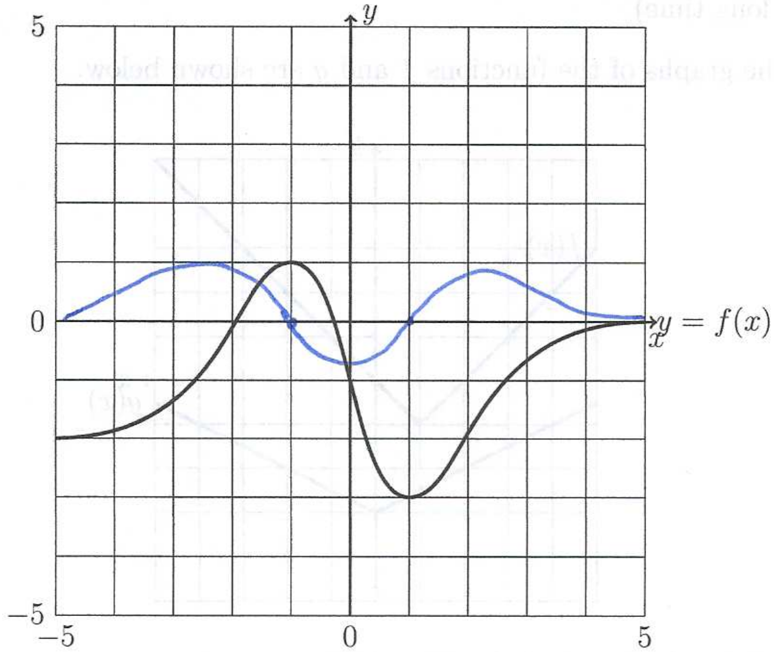
Q7 a) $f'(3) = \lim\limits_{h\to 0} \frac{f(3+h)-f(3)}{h} = \lim\limits_{h\to 0} \frac{\frac{1}{3+h-2} - \frac{1}{3-2}}{h} = \lim\limits_{h\to 0} \frac{1}{h}\left(\frac{1}{1+h} - 1\right)$

$= \lim\limits_{h\to 0} \frac{1}{h} \frac{1-1-h}{1+h} = \lim\limits_{h\to 0} \frac{-1}{1+h} = -1$

b) $f'(3) = \lim\limits_{h\to 0} \frac{f(3+h)-f(3)}{h} = \lim\limits_{h\to 0} \frac{\frac{1}{\sqrt{3+h-1}} - \frac{1}{\sqrt{3-1}}}{h} = \lim\limits_{h\to 0} \frac{1}{h}\left(\frac{1}{\sqrt{4+h}} - \frac{1}{\sqrt{2}}\right)$

$= \lim\limits_{h\to 0} \frac{1}{h} \frac{\sqrt{2} - \sqrt{4+h}}{\sqrt{2}\sqrt{4+h}} \frac{\sqrt{2}+\sqrt{4+h}}{\sqrt{2}+\sqrt{4+h}} = \lim\limits_{h\to 0} \frac{1}{h} \frac{2-4-h}{\sqrt{2}\sqrt{4+h}(\sqrt{2}+\sqrt{4+h})} = \lim\limits_{h\to 0} \frac{-1}{\sqrt{2}\sqrt{4+h}(\sqrt{2}+\sqrt{4+h})}$

(6) Consider the function $f(x)$ defined by the following graph.



(a) Label all regions where $f'(x) < 0$. $(-1,1)$
(b) Label all regions where $f'(x) > 0$. $(-5,1) \cup (1,5)$
(c) Sketch a graph of $f'(x)$ on the figure.
(d) What is $\lim_{x\to\infty} f(x)$? 0
(e) What is $\lim_{x\to-\infty} f'(x)$? 0

(7) Use the limit definition of the derivative to evaluate $f'(3)$, where
(a) $f(x) = \dfrac{1}{x-2}$
(b) $f(x) = \dfrac{1}{\sqrt{x-1}}$

(8) Find the derivatives of the following functions

(a) $3x^2 - 2x^3 - \sqrt{x^3} - 2\sqrt[3]{1/x^2}$
(b) $3x^4e^x$
(c) $\dfrac{3x-2}{3-2x}$
(d) $\dfrac{\sqrt{3x}-2}{1-\cos(x)}$
(e) $\tan(x)$
(f) $\sin^2(x)$
(g) $x^4e^{-2x^4}$
(h) $\dfrac{\sqrt{1-2x}}{4-\sin(3x)}$
(i) x^{x^2}
(j) $\sqrt{\csc(\ln(x))}$
(k) $\sin^{-1}(3x-2)$

$$= \frac{-1}{\sqrt{2}.\,2(\sqrt{2}+2)} = \frac{-1}{4+4+\sqrt{2}}.$$

Q8 a) $6x - 6x^2 - \frac{3}{2}x^{1/2} - 2.-\frac{2}{3}x^{-5/3}$

b) $12x^3e^x + 3x^4e^x$

c) $\frac{(3-2x)(3x-2)' - (3x-2)(3-2x)'}{(3-2x)^2} = \frac{9-6x+6x-4}{(3-2x)^2} = \frac{5}{(3-2x)^2} = .5(3-2x)^{-2}$

d) $\frac{(1-\cos(x))(\sqrt{3x}-2)' - (\sqrt{3x}-2)(1-\cos x)'}{(1-\cos(x))^2} = \frac{(1-\cos x)(\sqrt{3}\frac{1}{2}x^{-1/2}) - \sin x(\sqrt{3x}-2)}{(1-\cos(x))^2}.$

e) $(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x(\sin x)' - \sin x(\cos x)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$

f) $(\sin^2(x))' = 2\sin x.\cos x. = \sin 2x$

g) $4x^3e^{-2x^4} + x^4.e^{-2x^4}.-8x^3$

h) $\frac{(4-\sin(3x))(\sqrt{1-2x})' - (\sqrt{1-2x})(4-\sin(3x))'}{(4-\sin(3x))^2} = \frac{(4-\sin 3x)(\frac{1}{2}(1-2x)^{-1/2}.-2) - (\sqrt{1-2x})\,{-3\cos(3x)}}{(4-\sin(3x))^2}.$

i) $(x^{x^2})' = (e^{x^2\ln(x)})' = e^{x^2\ln(x)}\left(2x\ln(x) + x^2.\frac{1}{x}\right).$

j) $(\operatorname{cosec}(\ln(x))^{1/2})' = \cancel{t\,\operatorname{cosec}(\ln(x}$

$= \frac{1}{2}(\operatorname{cosec}(\ln(x)))^{-1/2}.(-\operatorname{cosec}(\ln(x)).\cot(\ln(x))).\frac{1}{x}$

k) $\frac{1}{\sqrt{1-(3x-2)^2}}.3.$

Q9 a) $6 - 12x - \frac{3}{4}x^{-1/2} - \frac{20}{9}x^{-8/3}$

b) $36x^2e^x + 12x^3e^x + 12x^3e^x + 3x^4e^x$

c) $-10(3-2x)^{-3} \cdot (-2)$.

e) $(\sec^2 x)' = 2\sec x \cdot \sec x \tan x = 2\sec^2 x \tan x$.

f) $\cos(2x) \cdot 2$

g) $12x^2 e^{-2x^4} + 4x^3 e^{-2x^4} \cdot -8x^3 - 56x^6 e^{-2x^4} + 8x^7 e^{-2x^4} \cdot 8x$.

Q10 a) $h'(2) = f(2)g'(2) + f'(2)g(2) = -1$

$2 \cdot \frac{1}{2} + 1 \cdot -2$

b) $h'(4) = \dfrac{g(4)f'(4) - f(4)g'(4)}{g(4)^2} = \dfrac{(-1)(-1) - 2(-\frac{1}{2})}{(-1)^2} = 2$

c) $h'(1) = f'(g(1)) \cdot g'(1) = f'(-2\tfrac{1}{2}) \cdot \frac{1}{2} = (-1) \cdot \frac{1}{2} = -\frac{1}{2}$.