Sample final solutions

Q1 a) $10x^4 - \frac{9}{4}x^{-1/4} - \operatorname{cosec}(3x)\cot(3x).3$

b) $$\frac{(x-x^2)\frac{1}{3-2x}.2 - (2x-x)\ln(3-2x)}{(x-x^2)^2}$$

c) $e^{-2x}.(-2)\cos(2-3x) + e^{-2x}.-\sin(2-3x)(-3).$

d) $\frac{1}{3}\left(e^{-\sin(2x)}+3\right)^{-2/3}.\ e^{-\sin(2x)}.-\cos(2x).2$

Q2 a) $-x^{-3} + 2\sin(x) - e^x + c$

b) $\int \frac{9-6x+4x^2}{x^{3/2}}\,dx = \int 9x^{-3/2} - 6x^{-1/2} + 4x^{1/2}\,dx$

$= 9x^{-1/2}.-2 - 6x^{1/2}.2 + 4x^{3/2}.\frac{2}{3} + c$

c) $\int_0^{\pi/6} \cos(3x)\sin^4(3x)\,dx$ $u = \sin(3x)$, $\frac{du}{dx} = 3\cos(3x)$

$\int_0^{1/\sqrt{2}} \cos(3x)\, u^4 \frac{dx}{du}\,du = \int_0^{1/\sqrt{2}} \cos(3x)\, u^4 \frac{1}{3\cos(3x)}\,du = \int_0^{1/\sqrt{2}} \frac{1}{3}u^4\,du$

$= \left[\frac{1}{15}u^5\right]_0^{1/\sqrt{2}} = \frac{1}{15.\cancel{16}\sqrt{2}}$

d) $\int \frac{1}{4+x^2}\,dx = \frac{1}{4}\int \frac{1}{1+(x/2)^2}\,dx$ $u = x/2$, $\frac{du}{dx} = 1/2$ $\frac{1}{4}\int \frac{1}{1+u^2}\frac{dx}{du}\,du$

$\frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\tan^{-1}(u) + c = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

Q1

a) $\overset{l'H}{=} \lim_{x\to 3} \dfrac{2x+1}{1} = 7$

b) $\overset{l'H}{=} \lim_{x\to 0} \dfrac{\cos(3x)\cdot 3}{-5e^{5x}} = \dfrac{3}{-5}$

c) $= \lim_{x\to 0+} e^{\ln(x)\sin(x)}$ (*) $\quad \lim_{x\to 0+} \ln(x)\sin(x) = \lim_{x\to 0+} \dfrac{\ln(x)}{1/\sin(x)}$

$\overset{l'H}{=} \lim_{x\to 0+} \dfrac{1/x}{-(\sin(x))^{-2}\cdot\cos(x)} = \lim_{x\to 0+} \dfrac{-\sin^2(x)}{x\cos(x)} \overset{l'H}{=} \lim_{x\to 0+} \dfrac{-2\sin(x)\cos(x)}{\cos(x) + x\cdot -\sin(x)} = 0$

so (*) $= \cancel{\lim}\ e^0 = \cancel{e}\ 1.$

d) $= \lim_{x\to 0} \dfrac{x^2 - \sin^2 x}{x^2\sin^2 x} \overset{l'H}{=} \lim_{x\to 0} \dfrac{2x - \overbrace{2\sin x\cos x}^{\sin 2x}}{2x\sin^2 x + x^2\, 2\sin x\cos x}$

$\overset{l'H}{=} \cancel{\lim_{x\to 0} \dfrac{2 - 2\cos^2 x + 2\sin^2 x}{\ }} = \lim_{x\to 0} \dfrac{2x - \sin 2x}{2x\sin^2 x + x^2\sin 2x}$

$\overset{l'H}{=} \lim_{x\to 0} \dfrac{2 - 2\cos 2x}{2\sin^2 x + 2x\underbrace{2\sin x\cos x}_{\sin 2x} + 2x\sin 2x + x^2\, 2\cos 2x}.$

$\overset{l'H}{=} \lim_{x\to 0} \dfrac{-4\sin 2x}{\underbrace{4\sin x\cos x}_{2\sin 2x} + 2\sin 2x + 4x\cos 2x + 2\sin 2x + 2x\cos 2x\cdot 2 + 2x\, 2\cos 2x - 4x^2\sin 2x}$

$= \lim_{x\to 0} \dfrac{-4\sin 2x}{\underset{8}{\cancel{4}}\sin 2x + \cancel{4\sin 2x} + \overset{10}{\cancel{8}}x\cos 2x + \cancel{4x\cos} - 4x^2\sin 2x}$

$\overset{l'H}{=} \lim_{x\to 0} \dfrac{-8\sin 2x}{16\cos 2x + \underset{8}{\cancel{10}}\cos 2x + 10x\cdot -2\sin 2x - 16x\sin 2x - 4x^2\cdot 2\cos 2x} = \dfrac{-8}{24} = -\dfrac{1}{3}$

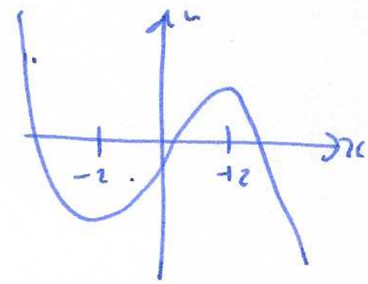
Q4 a) $f(x) = 12x - x^3$ $\quad f'(x) = 12 - 3x^2$ $\quad$ critical points, solve $f'(x) = 0$

$12 - 3x^2 = 0 \quad x^2 = 4 \quad x = \pm 2$

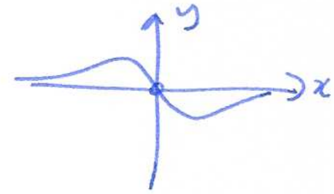
b) increasing on $(-2, 2)$

c) $f''(x) = -9x$ $\quad$ concave up ~~$(0, \infty)$~~ $(-\infty, 0)$
concave down $(0, \infty)$

d) $x = +2$ local max ~~min~~ $\quad x = -2$ local min $\quad$ e)

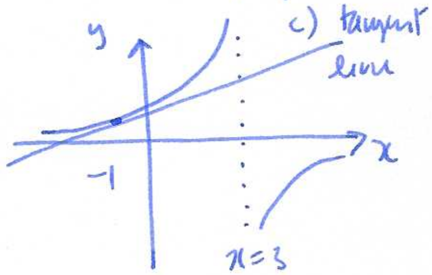


Q5 a) ~~$(0, 5)$~~ $(-\infty, -\frac{2}{3}) \cup (\frac{2}{3}, \infty)$ $\quad$ b) $(-\infty, 0)$ $\quad$ c)



Q6 $f(x) = \dfrac{3}{3-x}$

a) vertical asymptote at $x = 3$ $\quad \lim\limits_{x \to \infty} \dfrac{3}{3-x} = 0 \quad \lim\limits_{x \to -\infty} \dfrac{3}{3-x} = 0$



b) $f'(x) = -3(3-x)^{-2} \cdot (-1)$

$f'(-1) = 3/4^2 = 3/16 \quad f(-1) = \frac{3}{4}$

tangent line: $\quad y - 3/4 = 3/16\,(x+1)$

Q7 $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{(\cancel{x}+h) - \frac{1}{x+h} - \cancel{x} + \frac{1}{x}}{h}$

$= \lim\limits_{h \to 0} 1 + \frac{1}{h}\left(-\frac{1}{x+h} + \frac{1}{x}\right) = 1 + \lim\limits_{h \to 0} \frac{1}{h} \dfrac{-\cancel{x} + \cancel{x} + h}{x(x+h)} = 1 + \lim\limits_{h \to 0} \dfrac{1}{x(x+h)} =$

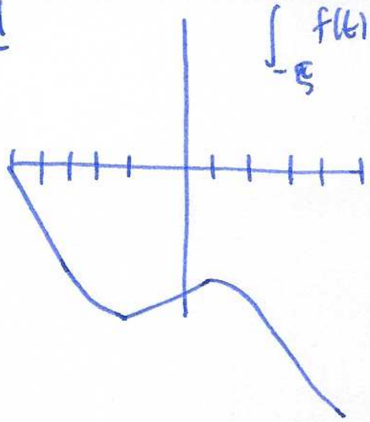
$1 + \frac{1}{x^2}$.

Q8 $x^3 y + 2xy^2 + 4x = 4 \rightsquigarrow 3x^2 y + x^3 y' + 2y^2 + 2x \cdot 2yy' + 4 = 0$

at $(2, -1)$: $-24y + 8y' + 2 - 8y' + 4 = 0 \leftarrow y' = \infty$ at $(2, -1)$

so tangent line vertical.

Q9 $\int_{-5}^{x} f(t)\,dt$



Q10

$$\int_{-2}^{1} 5 - x^2\,dx$$

$$= \left[5x - \tfrac{1}{3}x^3\right]_{-2}^{1}$$

$$= 5 - \tfrac{1}{3} - \left(-10 + \tfrac{8}{3}\right) = 15 - 3 = 12$$

Q11 $V = \frac{4}{3}\pi r^3$ $\qquad A = 4\pi r^2$.

$$\underbrace{\frac{dV}{dt}}_{3} = 4\pi \underbrace{r^2}_{2} \frac{dr}{dt} \qquad \frac{dA}{dt} = 8\pi r \frac{dr}{dt}$$

$$3 = 16\pi \frac{dr}{dt} \qquad \frac{dr}{dt} = \frac{3}{16\pi}. \qquad \frac{dA}{dt} = 8\pi . 2 . \frac{3}{16\pi} = 3 \text{ in}^2/\text{s}.$$

Q12 $f(x) = x^{1/3}$ $\qquad f(27) = 3$.

$f'(x) = \frac{1}{3}x^{-2/3}$.

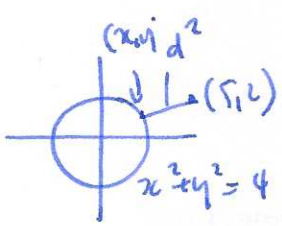
$$f(25) \approx f(27) - 2f'(27)$$

$$= 3 - 2 . \frac{1}{3}\frac{1}{9} = 3 - \frac{2}{27}$$

absolute error: $\left|\sqrt[3]{25} - (3 - 2/27)\right| \approx 0.00191$

percentage error $\dfrac{\text{abs error}}{\sqrt[3]{25}} \times 100 \approx 0.06530\%$

Q13



$$d^2 = (x-5)^2 + (y-2)^2 = (x-5)^2 + \left(\sqrt{4-x^2} - 2\right)^2$$

$$\frac{d(d^2)}{dx} = 2(x-5) + 2\left(\sqrt{4-x^2} - 2\right) . \tfrac{1}{2}(4-x^2)^{-1/2} . -2x$$

critical pt: $\dfrac{d(d^2)}{dx} = 0$: $\quad x - 5 - \dfrac{\left(\sqrt{4-x^2} - 2\right)}{\sqrt{4-x^2}} . x = x - 5 - x + \dfrac{2x}{\sqrt{4-x^2}} = 0$

$2x = 5\sqrt{4-x^2}$ $\qquad 4x^2 = 25(4-x^2)$ $\qquad 29x^2 = 25.4$ $\qquad x = \frac{10}{\sqrt{29}}$