Math 301 Introduction to Proof Fall 22 Sample Midterm 2

- (1) Show that $\sqrt[3]{2}$ is irrational.
- (2) Find an example of a function $f \colon \mathbb{N} \to \mathbb{Z}$ which is
 - (a) injective but not surjective
 - (b) neither injective nor surjective
- (3) Find an example of a function $f \colon \mathbb{R} \to \mathbb{R}$ which is
 - (a) surjective but not injective
 - (b) neither injective nor surjective
- (4) Find explicit bijections between the following subintervals of \mathbb{R} .
 - (a) (0, 1) and $(1, \infty)$
 - (b) $(1,\infty)$ and $(0,\infty)$
 - (c) $(0,\infty)$ and \mathbb{R}
 - (d) (0,1) and \mathbb{R}
- (5) Say a set A is *countable* if there is an injective map $f: A \to \mathbb{N}$. Show that the product of countable sets is countable. Show that the set of quadratic integers is countable. (A number is a quadratic integer if it is the solution of a quadratic equation with integer coefficients and leading coefficient equal to 1.)
- (6) What do the following statements mean in ordinary language? Write out their negations using quantifiers.
 - (a) $(\forall n \in N) \ (x \neq 2^n)$
 - (b) $(\exists a, b \in \mathbb{N})$ $(\pi = a/b)$
 - (c) $(\forall b \in B) (\exists a \in A) (f(a) = b)$, assume $f: A \to B$
 - (d) $(\forall n \in \mathbb{N}) \ (\exists m \in \mathbb{N}) \ (n^2 n = 2m)$
 - (e) $(\exists A \subset \mathbb{R}) \ (\forall x \in \mathbb{R}) \ (\exists a \in A)(x > a)$
 - (f) $(\forall n \in \mathbb{N}) \ (\exists p \in \mathbb{N}) \ ((p > n) \text{ and } [(\forall q \in \mathbb{N}) \ (q | p \Rightarrow (q = 1) \text{ or } q = p)])$
- (7) State the negation of the following statements, using appropriate quantifiers.
 (a) √π is irrational.
 - (b) The function f is surjective but not injective.
 - (c) The integer n is divisible by two distinct primes.
 - (d) There is a surjective function $f: A \to B$.
 - (e) The function $f \colon \mathbb{R} \to \mathbb{R}$ is bounded below.
 - (f) $\lim_{x \to 1} f(x) = 2.$

- (8) Consider the statement: If $S = \emptyset$ then $\mathcal{P}(S) = \emptyset$.
 - (a) Is the statement true or false?
 - (b) State the contrapositive. Is it true or false?
 - (c) State the converse. Is it true or false?
 - (d) State the contrapositive of the converse. Is it true or false?
 - (e) Consider the statement: If $S = \emptyset$ then $\mathcal{P}(S) \neq \emptyset$. Is this statement true or false? Is this statement equivalent to any of the statements above?
- (9) Write out careful proofs, or give counterexamples, to the following statements.
 - (a) Show that the sum of the squares of any three consecutive integers, plus one, is divisible by 3.
 - (b) Show that the sum of any four consecutive integers is divisible by 4.
 - (c) A function $f: A \to B$ has an inverse if and only if it is both surjective and injective.
 - (d) If f is not surjective, then it is injective.
 - (e) If g is surjective, then $f \circ g$ is surjective.
 - (f) If g and $f \circ g$ are injective, then g is injective.
 - (g) If $f \circ g \colon \mathbb{R} \to \mathbb{R}$ is decreasing, and $g \colon \mathbb{R} \to \mathbb{R}$ is decreasing, then f is decreasing.
 - (h) If $x \in \mathbb{R}$ and $x^2 \leq x$, then $x \leq 1$.
- (10) Let $f: X \to Y$ be a function. For any subset $A \subseteq Y$, let $f^{-1}(A)$ be the pre-image of A in X. Show that this defines a function from $\mathcal{P}(Y)$ to $\mathcal{P}(X)$. Can you say when it is injective or surjective?
- (11) Suppose that $f: X \to Y$ and let $A \subseteq X$ and $B \subseteq Y$.
 - (a) Prove or give a counterexample: $f^{-1}(f(A)) \subseteq A$
 - (b) Prove or give a counterexample: $B \subseteq f(f^{-1}(B))$
 - (c) Prove or give a counterexample: $f(A \cup f^{-1}(B)) = f(A) \cup B$.
 - (d) Prove or give a counterexample: $f^{-1}(f(A) \cap B) = A \cap f^{-1}(B)$