SMT2 Solutions

Q1 Theorem $\sqrt[3]{2}$ is irrational.

Proof (by contradiction) suppose $\sqrt[3]{2} = \frac{a}{b}$ with fraction in lowest terms. Then $2 = \frac{a^3}{b^3} \Rightarrow 2b^3 = a^3$, so $2 | a^3$

Lemma if $2|n^3$ then $2|n$.

Proof (contrapositive) we will show if $2 \nmid n$ then $2 \nmid n^3$.

If $2 \nmid n$ then n is odd, so $n = 2k+1$ for some integer k. Therefore $n^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1$, which is odd, so $2 \nmid n^3$. $\square$ (Lemma).

We have shown $2|a^3$, so Lemma implies $2|a$, so $a = 2c$ for some integer c. Therefore $a^3 = 8c^3$, so $2b^3 = 8c^3 \Rightarrow b^3 = 4c^3 \Rightarrow 2|b^3$, but then Lemma $\Rightarrow 2|b$, so $2|a$ and $2|b$ and they have a common factor ※. This is a contradiction, so $\sqrt[3]{2}$ is irrational $\square$.

Q2 $f: \mathbb{N} \to \mathbb{Z}$ a) injective but not surjective $n \mapsto 2n$

b) neither injective nor surjective: $\begin{cases} n \mapsto n & \text{if } n \text{ even} \\ n \mapsto n+1 & \text{if } n \text{ odd} \end{cases}$

Q3 $f: \mathbb{R} \to \mathbb{R}$ a) surjective but not injective: $f(x) = x(x-1)(x+1)$

b) neither injective nor surjective: $\sin(x)$

Q4 a) $x \mapsto \frac{1}{x}$ b) $x \mapsto 1-x$ c) $x \mapsto \log(x)$ d) $x \mapsto \log(1-\frac{1}{x})$

Q5 Thm The product of countable sets is countable.

Proof Let A, B be countable sets, so there are injective maps $f: A \to \mathbb{N}$ and $g: B \to \mathbb{N}$. Claim the map $A \times B \xrightarrow{f \times g} \mathbb{N} \times \mathbb{N}$ is given by $(a,b) \mapsto (f(a), g(b))$ is injective. Proof (of claim) suppose ~~f(a)~~ $f((a,b)) = f((c,d))$, then $(f(a), g(b)) = (f(c), g(d))$ ~~as f is inj.~~ so $f(a) = f(c)$ and $g(b) = g(d)$. As f, g are injective, this implies $a = c$ and $b = d$, so $(a,b) = (c,d)$ and $f \times g$ is injective $\square$. Claim there is an injective map $\mathbb{N} \times \mathbb{N} \xrightarrow{p} \mathbb{N}$

Proof send $(a,b) \mapsto 2^a 3^b$. Suppose $p(a,b) = p(c,d)$, then $2^a 3^b = 2^c 3^d$. By unique prime factorization, this implies $a = c$ and $b = d$, so p is injective. $\square$. The composition of injective maps is injective, so the map $A\times B \xrightarrow{f\times g} \mathbb{N}\times\mathbb{N} \xrightarrow{p} \mathbb{N}$ is injective, so $A\times B$ is countable $\square$. ②

Q6 a) x is not a power of 2. negation: $(\exists n\in\mathbb{N})(x=2^n)$.

b) π is rational. negation: $(\forall a,b\in\mathbb{N})(\pi\neq a/b)$

c) $(\forall b\in B)(\exists a\in A)(f(a)=b)$ f is surjective. negation: $(\exists b\in B)(\forall a\in A)(f(a)\neq b)$

d) for all ~~integers~~ natural numbers n, n^2-n is even. negation: $(\exists n\in\mathbb{N})(\forall m\in\mathbb{N})(n^2-n\neq 2m)$

e) there is a subset of $\mathbb{R}$ which is not bounded below. negation: $(\forall A\subseteq\mathbb{R})(\exists x\in\mathbb{R})(\forall a\in A)(x\leq a)$

f) the set of prime numbers is not bounded above. negation: $(\exists n\in\mathbb{N})(\forall p\in\mathbb{N})((p\leq n)$ or $[(\exists q\in\mathbb{N})(q|p$ and $(q\neq 1)$ and $(q\neq p))])$.

Q7 a) $(\forall a,b\in\mathbb{N})(\sqrt{\pi}\neq a/b)$

b) (assume $f:A\to B$): $(\forall b\in B)(\exists a\in A)(f(a)=b)$ and $(\exists a,b\in A)(f(a)=f(b)$ and $a\neq b)$.

c) $(\exists p,q\in\mathbb{N})$ $(p\neq q)$ and $(p|n)$ and $(q|n)$ and $(\forall a\in\mathbb{N})(a|p)\Rightarrow[(a=1)$ or $(a=p)]$ and $(\forall a\in\mathbb{N})(a|q)\Rightarrow[(a=1)$ or $(a=q)]$

d) $(\exists f:A\to B)(\forall b\in B)(\exists a\in A)(f(a)=b))$.

e) $(\exists x\in\mathbb{R})(\forall y\in\mathbb{R})(f(y)>x)$.

f) $(\forall\epsilon>0)(\exists\delta>0)(|x-1|<\delta \Rightarrow |f(x)-2|<\epsilon)$

Q8 a) false b) $P(S)\neq\phi \Rightarrow S\neq\phi$ false c) $P(S)=\phi\Rightarrow S=\phi$ (vacuously) true d) $S\neq\phi\Rightarrow P(S)\neq\phi$ true. e) true, not equivalent to any of the above.

Q9 a) ~~Thm for any $n\in\mathbb{Z}$, $3|n^2+(n+1)^2+(n+2)^2$~~

~~Proof consider $n^2+(n+1)^2+(n+2)^2 = n^2+n^2+2n+1+n^2+4n+4 = 3n^2+$~~

False: counterexample $n=1$: $1^2+2^2+3^2=1+4+9=14$ $3\nmid 14$.

b) False: counterexample $n=1$: $1+2+3+4=10$ $4\nmid 10$.

c) Thm $f: A \to B$ has an inverse iff it is both surjective and injective. ③

Proof $\Rightarrow$ suppose $f: A \to B$ has an inverse function $f^{-1}: B \to A$.

claim: f is surjective. proof for any $b \in B$, $f^{-1}(b) \in A$ and $f(f^{-1}(b)) = b$ □

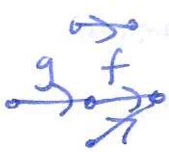
claim: f is injective proof suppose $f(a) = f(b)$, then $f^{-1}(f(a)) = f^{-1}(f(b))$ $\Rightarrow$ $a = b$ □.

$\Leftarrow$ claim: define $f^{-1}(b) = f^{-1}(\{b\})$ for all $b \in B$. claim this defines a function. check: ① $f^{-1}(\{b\})$ is non-empty for all $b \in B$: Proof as f is surjective, for all $b \in B$, $\exists\, a \in A$ s.t. $f(a) = b$, so $a \in f^{-1}(\{b\})$, so $f^{-1}(\{b\}) \neq \phi$ □. ② $f^{-1}(\{b\})$ contains exactly one element. Proof suppose $x \neq y$ $x, y \in f^{-1}(\{b\})$ then $f(x) = f(y) = b$. As f is injective, this implies $x = y$ ※ so $f^{-1}(\{b\})$ has exactly one element, and can define $f^{-1}: B \to A$ by $f^{-1}(b) = f^{-1}(\{b\})$ □.

d) False :

e) false :

f) false :



g) false : $g(x) = -x$ $f(x) = 2x$.

h) Thm If $x^2 \leq x$ then $x \leq 1$

Proof (contrapositive) show: if $x > 1$ then $x^2 > x$. Consider $x > 1$ if $x > 0$ then $x \times x > 1 \times x \Rightarrow x^2 > x$ □.

Q10 Claim $A \mapsto f^{-1}(A)$ defines a function $f^{-1}: P(Y) \to P(X)$

Proof: ① for any $A \in P(Y)$, $A \subseteq Y$, $f^{-1}(A) \subseteq X$, so $f^{-1}(A) \in P(X)$, as required.

② $f^{-1}(A)$ is well defined: let B, C be $f^{-1}(A)$. If $x \in B$, then $f(x) \in A$, so $x \in C$. Similarly, if $x \in C$, then $f(x) \in A$, so $x \in B$, therefore $B = C$ □.

~~claim: f^{-1} injective. Proof Let $A, B \subseteq Y$ s.t. $f^{-1}(A) = f^{-1}(B)$. Suppose $a \in A$,~~

Thm f^{-1} injective iff f surjective.

<u>Proof</u> $\Rightarrow$ (contrapositive) suppose f not surjective, then $\exists\, y \in Y$ s.t. $\forall x \in X$ $f(x) \neq y$. Then $f^{-1}(\{y\}) = \emptyset$, but $f^{-1}(\emptyset) = \emptyset$ and $\{y\} \neq \emptyset$, so f^{-1} not injective $\square$

$\Leftarrow$ claim: if f surjective, $f(f^{-1}(A)) = A$. Proof (of claim) If $A = \emptyset$, then $f^{-1}(\emptyset) = \emptyset$, so $f(\emptyset) = \emptyset$, as required. Assume $A \neq \emptyset$. For every $a \in A$, there is an $x \in X$ s.t. $f(x) = a$, and thus $x \in f^{-1}(A)$ by definition of $f^{-1}(A)$. As $f(x) = a$ this implies $a \in f(f^{-1}(A))$, so $A \subseteq f(f^{-1}(A))$. Now consider $y \in f(f^{-1}(A))$. So there is $x \in f^{-1}(A)$ s.t. $f(x) = y$. But $x \in f^{-1}(A)$ means $f(x) \in A$, so $y = f(x) \in A$, and $f(f^{-1}(A)) \subseteq A$, as required $\square$ claim. Finally, suppose $f^{-1}(A) = f^{-1}(B)$, then $f(f^{-1}(A)) = f(f^{-1}(B))$, but by claim, $f(f^{-1}(A)) = A = f(f^{-1}(B)) = B$, so $A = B$, so f^{-1} is injective. $\square$.

<u>Thm</u> f^{-1} surjective iff f injective.

<u>Proof</u> $\Rightarrow$ (contrapositive) suppose f not injective, then there is $x \neq y \in X$ s.t. $f(x) = f(y)$ <u>claim</u> $\{x\}$ does not lie in image of f^{-1}. Proof (of claim). Suppose $x \in f^{-1}(A)$, then $f(x) \in A$. As $f(x) = f(y)$, this implies $f(y) \in A$, so $y \in f^{-1}(A)$, so $\{x, y\} \subseteq f^{-1}(A)$, so $f^{-1}(A) \neq \{x\}$ for all $A \subseteq Y$ $\square$.

$\Leftarrow$ <u>claim</u> if f injective, then $f^{-1}(f(A)) = A$. Proof (of claim) First show $A \subseteq f^{-1}(f(A))$. Suppose $a \in A$, then $f(a) \in f(A)$, and so $a \in f^{-1}(f(A))$. Therefore $A \subseteq f^{-1}(f(A))$. Now show $f^{-1}(f(A)) \subseteq A$. Suppose not, then there is $x \in X$ such that $x \in f^{-1}(f(A))$ but $x \notin A$. But $f(x) \in f(A)$, so $\exists$ $a \in A$ s.t. $f(a) = f(x)$, but $a \neq x$ as $a \in A$ and $x \notin A \Rightarrow f$ not injective ↯ $\square$claim. Finally, suppose $A \subseteq X$. Then $f^{-1}(f(A)) = A$, so $f(A)$ gets mapped to A by f^{-1}, so f^{-1} surjective $\square$.

<u>Q11</u> a) counterexample:
$$\begin{tikzcd} 1 \arrow[r] & 3 \\ 2 \arrow[ur] & 4 \end{tikzcd}$$
$A = \{1\}$ $\quad f^{-1}(f(A)) = \{1,2\} \neq \{1\}$.

b) counterexample: $B \subseteq f(f^{-1}(B))$ $\quad B = \{4\}$ $\quad f^{-1}(\{4\}) = \emptyset$, $f(\emptyset) = \emptyset$ $\quad \{4\} \neq \emptyset$.

c) counterexample $A = \emptyset$, $B = \{4\}$. $f(A \cup f^{-1}(B)) = f(\emptyset) = \emptyset \neq f(\emptyset) \cup \{4\} = \emptyset \cup \{4\} = \{4\}$

d) counterexample $A = \{1\}$, $B = \{3\}$. $f^{-1}(f(A) \cap B) = f^{-1}(\{3\}) = \{1,2\} \neq \{1\} \cap \{1,2\} = \{1\}$.