Math 301 Introduction to Proof Fall 22 Midterm 1

Name: ______________________________

- **Start each question on a fresh sheet of paper. Staple together in numerical order at the end of the exam.**

(1) For each of the following statements, find two distinct elements in the truth set, and two distinct elements not in the truth set. (Indicate clearly which are which.)
 (a) $x + y = 1$, where the universe is $\mathbb{R} \times \mathbb{R}$.
 (b) $A \subseteq B$, where the universe is all subsets of $\mathbb{Z}$.

(2) Consider the statement:
 If x and y are real numbers with $x > y$ then $(x+y)^2 > x^2 + y^2$.
 Which, if any, of the following substitutions give a counter example.
 (a) $x = 1, y = 2$ (b) $x = -2, y = 1$ (c) $x = 1, y = -2$ (d) $x = 2, y = 1$

(3) Write out a careful proof of the fact that the product of any two odd number is odd.

(4) Prove or disprove the following statement: If $A \cup B = A \cup C$ then $B = C$.

(5) Prove or disprove the following statement: $A - (B \cap C) = (A - B) \cap (A - C)$.

(6) Prove or disprove the following statement: $A' \cap B = B - A$.

(7) Prove or disprove the following statement: If $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ then $A \subseteq B$.

(8) State which of the following statements, are true, vacuously true, or false.
 (a) For integers a, b and c, if $a \mid bc$ then $a \mid b$ or $a \mid c$.
 (b) If x is an integer with $2x = 1$ then x is positive.
 (c) If $A \subseteq C$ and $B \subseteq D$ then $A \cup B \subseteq C \cup D$.
 (d) If $\mathcal{P}(A) \cap \mathcal{P}(B) = \varnothing$ then $A \cap B = \varnothing$.

(9) Write out a careful proof of the fact that if $a \mid b$ then $a^2 \mid b^2$.

(10) Consider the following theorem and proof. Is the theorem correct? Is the proof correct? Explain.

Theorem. *For any sets A and B, $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$.*

Proof. Suppose $x \in \mathcal{P}(A \cup B)$. Then $x \subseteq A$ or $x \subseteq B$, so $x \in \mathcal{P}(A)$ or $x \in \mathcal{P}(B)$. Therefore $x \in \mathcal{P}(A) \cup P(B)$. $\square$

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MTI Solutions

Q1 a) $(0,1)$, $(1,0)$ in truth set, $(0,0)$, $(1,1)$ not in truth set.

b) $(\emptyset, \{1\})$ $(\emptyset, \mathbb{Z})$ in truth set, $(\{1\}, \emptyset)$ $(\mathbb{Z}, \emptyset)$ not in truth set.

Q2 c)

Q3 Thm The product of any two odd numbers is odd.

Proof Let x and y be odd numbers, then $x = 2a+1$ and $y = 2b+1$ for two integers a and b. Then $xy = (2a+1)(2b+1) = 4ab+2a+2b+1 = 2(2ab+a+b)+1$, which is odd, so xy is odd $\square$.

Q4 False: $A = \{1\}$, $B = \emptyset$, $C = \{1\}$ counterexample.

Q5 

$A-(B\cap C)$ 

$(A-B)\cap(A-C)$ False: $A = \{1\} = B$, $C = \emptyset$ counterexample.

Q6 Thm $A' \cap B = B - A$

Proof $\subseteq$: suppose $x \in A' \cap B$, then x is in B, but not in A, so $x \in B - A$.

$\supseteq$ suppose $x \in B - A$, then x is in B, but not in A, so x is in B and A', so $x \in A' \cap B$. $\square$.

Q7 Thm If $P(A) \subseteq P(B)$ then $A \subseteq B$.

Proof Assume $P(A) \subseteq P(B)$. Let $x \in A$, then $\{x\} \subseteq A$, so $\{x\} \in P(A)$. As $P(A) \subseteq P(B)$, this means $\{x\} \in P(B)$, so $\{x\} \subseteq B$, so $x \in B$. Therefore $A \subseteq B$. $\square$.

Q8 a) False b) vacuously true c) True d) vacuously true.

Q9 Thm If $a|b$ then $a^2|b^2$.

Proof If $a|b$ then $b = ac$ for some integer c.
Then $b^2 = b.b = acac = a^2c^2$, so $b^2 = a^2c^2$ for some integer c^2, so $a^2|b^2$. $\Box$.

Q10 Thm: false counterexample: $A = \{1\}$, $B = \{2\}$. $A \cup B = \{1,2\}$.
but $A \cup B \notin P(A) \cup P(B)$.

Proof $x \in P(A \cup B) \not\Rightarrow x \subseteq A$ or $x \subseteq B$; see counterexample.