Math 301 Introduction to Proof Fall 22 Sample Final

(1) Consider the statement

If x and y are real numbers with $x < y$ then $\frac{1}{x^2} > \frac{1}{y^2}$.

Which, if any, of the following substitions gives a counterexample:

(a) $x = 1, y = 2$ (b) $x = 2, y = 1$ (c) $x = -2, y = 1$

(2) State which of the following statements are true, vacuously true, or false.

(a) $A' \cap B' = (A \cap B)'$

(b) If $\mathcal{P}(A) = \varnothing$, then $A = \varnothing$

(c) $(A \cup B) - C = (A - C) \cup (B - C)$

(3) Let $f \colon X \to Y$ be a function, and let A and B be subsets of X. Prove or disprove: $f(A \cap B) = f(A) \cap f(B)$.

(4) Let $f \colon X \to Y$ be a function, and let A be a subset of X. Show that $A \subseteq f^{-1}(f(A))$. Is $f^{-1}(f(A)) = A$?

(5) Either give examples of functions with the properties below, or explain why they don't exist.

(a) An injective function from $[0, 1]$ to $\mathbb{Z}$

(b) An sujective function from $(0, 1)$ to $(0, \infty)$

(c) An injective function from $[0, 1] \times [0, 1]$ to $[0, 1]$

(6) State the negation of the following statements, using appropriate quantifiers.

(a) The function $f \colon A \to B$ is injective.

(b) There are no surjective functions $f \colon A \to B$.

(c) The set $A \subseteq \mathbb{R}$ is bounded below.

(7) Write out careful proofs, or give counterexamples, to the following statements.

(a) If n is an integer then $n^3 + n^2$ is even.

(b) If g and $g \circ f$ are injective, then g is injective.

(c) If $f \colon \mathbb{R} \to \mathbb{R}$ is increasing, and $g \colon \mathbb{R} \to \mathbb{R}$ is increasing, then the function defined by $x \mapsto f(x)g(x)$ is increasing.

(8) Write out the negation of the statement, "$(\forall x)(p(x)) \Rightarrow (\exists x)(\sim q(x))$".

(9) Let $f \colon A \to B$ be a function. Write out the converse to the statement "If f is not injective then it is surjective". Prove or disprove the converse.

(10) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. Write out the contrapositive to the statement "If the function f is not strictly increasing then it is decreasing.". Prove or disprove the contrapositive.

(11) Show that the following numbers are irrational: $\sqrt{3}, \sqrt{5}, \sqrt{15}, \sqrt{3} - \sqrt{5}$.

(12) Show that $1 \times 2 + 2 \times 2^2 + 3 \times 2^3 + \cdots + n \times 2^n = (n-1)2^{n+1} + 2$.

(13) Show that $n^3 \leq 3^n$ for $n \geq 3$.

(14) Use induction to show that $12^n - 7^n$ is divisible by 5.

(15) Show that $x^{2n} - y^{2n}$ is divisible by $x + y$ for all integers x, y and all natural numbers n.

(16) Show that $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} < 1$.

(17) Show that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$.

(18) Consider the relation on $\mathbb{R}$ determined by $x \sim y$ if xy is an integer. Is this an equivalence relation?

(19) Consider the relation on functions $f\colon \mathbb{R} \to \mathbb{R}$ given by $f \sim g$ if there are numbers $A, B \in \mathbb{R}$ such that for all x, $f(x) \leq Ag(x) + B$. Is this an equivalence relation?

(20) Define a relation on sets by $A \sim B$ if there is a surjection $f\colon A \to B$. Is this an equivalence relation on sets?

(21) Let F be the set of all functions $f\colon \mathbb{R} \to \mathbb{R}$. Define a relation on F by $f \sim g$ if $g = f'$. Is this an equivalence relation? Does it define a function?