Sample final solutions

Q1 c)

Q2 a) false b) vacuously true c) true

Q3 False $\begin{array}{l} 1 \to 3 \\ 2 \nearrow \; \cdot 4 \end{array}$ $A=\{1\}, B=\{2\}$ $f(A)=\{3\}, f(B)=\{3\}$ $f(A)\cap f(B)=\{3\}$.
$A\cap B=\phi$ $f(\phi)=\phi\neq\{3\}$.

Q4 $f: X\to Y$ $A\subseteq X$.
Thm $A\subseteq f^{-1}(f(A))$ Proof Let $x\in A$. Then $f(x)\in f(A)$, by definition of image of A. By definition of pre-image, as image of x under f is $f(x)$, $x\in f^{-1}(f(A))$, so $A\subseteq f^{-1}(f(A))$ $\Box$.
$f^{-1}(f(A))\neq A$ Example: $\begin{array}{l} 1 \to 3 \\ 2 \nearrow \; \cdot 4 \end{array}$ $A=\{1\}, f(A)=\{3\}$, $f^{-1}(f(A))=\{1,2\}\neq\{1\}$.

Q5 a) no such function as $[0,1]$ uncountable, and if $f:[0,1]\to\mathbb{Z}$ injective, then f gives bijection $[0,1]$ to $f([0,1])\subseteq\mathbb{Z}$, and any infinite subset of $\mathbb{Z}$ countable.
b) $f(x)=\frac{1}{1-x}-1$
c) $(0.a_1a_2a_3\ldots,\ 0.b_1b_2b_3\ldots)\mapsto(0.a_1b_1a_2b_2\ldots)$

Q6 a) $(\forall x,y\in A)(f(x)=f(y)\Rightarrow x=y)$.
b) $(\forall f:A\to B)(\exists b\in B)(\forall a\in A)(f(a)\neq b)$.
c) $(\exists x)(\forall a\in A)(x\leq a)$

Q7 a) Thm n^3+n^2 is even. Proof: $n^3+n^2=n.n(n+1)$. If n is even, then $n=2k$, so $2k(2k)(2k+1)$ is even. If n is odd, then $n+1$ is even, so $n+1=2k$, $(2k-1)^2 2k$, is even $\Box$.
b) False: $\begin{array}{l} \cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot \\ \qquad\quad \nearrow \end{array}$ f, $g\circ f$ injective but g not injective.
c) ~~Thm~~ False: $f(x)=x$ increasing, $g(x)=x$ increasing but $f(x)g(x)=x^2$, not increasing.

Q8 $(\forall x)(p(x))$ and $(\forall x)(q(x))$.

Q9 If f is surjective then it is not injective. False: example: $1\xrightarrow{f}2$

Q10 If f is not decreasing then it is strictly increasing

False: $f(x)=x^2$ not decreasing and not strictly increasing ($f:\mathbb{R}\to\mathbb{R}$).

Q11 Thm $\sqrt{3}$ is irrational. Proof (by contradiction) Suppose $\sqrt{3}=a/b$ in lowest terms, $a,b\in\mathbb{Z}$. Then $3=a^2/b^2 \Rightarrow 3b^2=a^2 \Rightarrow 3|a^2$. Claim: if $3|a^2$ then $3|a$. Proof (of claim) (contrapositive) If $3\nmid a$ then $a=3k+1$ or $a=3k+2$. Then $a^2=9k^2+6k+1$ or $a^2=9k^2+6k+4$ ← neither divisible by 3 $\#$ $3|a^2$. □(claim).

As $3|a$, $a=3k$ for some $k\in\mathbb{Z}$, so $3b^2=(3k)^2=9k^2 \Rightarrow b^2=3k^2$. Therefore $3|b^2$, so claim $\Rightarrow 3|b$. But then a,b have common factor $\#$ lowest terms. □

Thm $\sqrt{5}$ is irrational. Proof (by contradiction) Suppose $\sqrt{5}=a/b$ in lowest terms, $a,b\in\mathbb{Z}$. Then $5=a^2/b^2 \Rightarrow 5b^2=a^2$. Claim: If $5|a^2$ then $5|a$. Proof (uses unique prime factorization) $a=p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k}$ for distinct primes p_i, so $a^2=p_1^{2n_1}p_2^{2n_2}\cdots p_k^{2n_k}$. But if $5|a^2$, then some $p_i=5$ as 5 prime, so $a=\cancel{5^{n_i}}\,p_1^{n_1}\cdots(p_i=5)^{n_i}\cdots p_k^{n_k} \Rightarrow 5|a$. □

So $a=5k$ for some k, so $5b^2=(5k)^2=25k^2 \Rightarrow b^2=5k^2 \underset{\text{by claim}}{\Rightarrow} 5|b$. But then a,b have common factor $\#$ lowest terms □.

Thm $\sqrt{15}$ is irrational. Proof (by contradiction) Suppose $\sqrt{15}=a/b$ in lowest terms, $a,b\in\mathbb{Z}$. Then $15=a^2/b^2 \Rightarrow 15b^2=a^2$. But then $3|a^2 \Rightarrow 3|a$ by claim from $\sqrt{3}$ case. So $a=3k$, so $15b^2=(3k)^2 \Rightarrow 15b^2=9k^2 \Rightarrow 5b^2=3k^2 \Rightarrow 3|5b^2$. As $3\nmid 5 \Rightarrow 3|b^2$, so $3|b$ by claim, so a,b have common factor $\#$ □.

Thm $\sqrt{3}-\sqrt{5}$ is irrational. Proof (by contradiction) Suppose $\sqrt{3}-\sqrt{5}=a/b$, $a,b\in\mathbb{Z}$. Then $(\sqrt{3}-\sqrt{5})^2=a^2/b^2 \Rightarrow 3-2\sqrt{15}+5=a^2/b^2 \Rightarrow \sqrt{15}=\dfrac{8-a^2/b^2}{2}\in\mathbb{Q}$ $\#$ $\sqrt{15}$ irrational □.

Q12 Thm $1\times2+2\times2^2+3\times2^3+\cdots+n\times2^n=(n-1)2^{n+1}+2$.

Proof (induction) Base case: $n=1$: $1\times2=(1-1)2^{n+1}+2=0+2=2$ ✓. □

Induction step: assume true for n. Consider $\underbrace{1\times2+2\times2^2+\cdots+n\times2^n}_{=(n-1)2^{n+1}+2 \text{ by induction hypothesis } P(n)}+(n+1)\times2^{n+1}$

$=(n-1)\times2^{n+1}+2+(n+1)\times2^{n+1}=2n\times2^{n+1}+2=n\times2^{(n+1)+1}+2$

Therefore $1\times2+\cdots+(n+1)\times2^{n+1}=(n)\times2^{n+2}+2$, as required □ result follows by induction □.

Q13 Thm $n^3\leq 3^n$ for $n\geq3$. Proof (induction) Base case $n=3$: $3^3=3^3$ ✓ □.

Induction step: Assume true for n: $n^3\leq3^n$. Now consider $(n+1)^3=n^3+3n^2+3n+1$

~~Observe that~~ Claim: $3n^2+3n+1\leq 2n^3$. Proof (of claim) For $n\geq1$, $3n^2+3n+1\leq 7n^2$

If $n \in \mathbb{Z}$ and $n \geq 3$, then $n \geq 7/2$, combine: $n^2 \leq n^2$ and $\frac{7}{2} \leq n$ $\Rightarrow$ $\frac{7}{2}n^2 \leq n^3 \Rightarrow 7n^2 \leq 2n^3$. Therefore $3n^2+3n+1 \leq 7n^2 \leq 2n^3$ for $n \geq 3$. □.

Therefore $(n+1)^3 = n^3+3n^2+3n+1 \leq n^3+2n^3 = 3n^3 \leq 3 \cdot 3^n = 3^{n+1}$, so $p(n+1)$ follows □.

So result follows by induction. □.

Q14 Thm $5 \mid 12^n - 7^n$ Proof (induction) Base case: $n=1$ $5 \mid 12-7=5$ ✓ □.

Induction step: consider $12^{n+1}-7^{n+1} = 12 \cdot 12^n - 7 \cdot 7^n = (5+7)12^n - 7 \cdot 7^n =$

$5 \cdot 12^n + 7(12^n - 7^n)$ By induction hypothesis $5 \mid 12^n-7^n$, so $5 \mid 5 \cdot 12^n + 7 \cdot (12^n - 7^n)$ □.

Q15 Thm $\forall x,y,n \in \mathbb{Z}$, $x+y \mid x^{2n}-y^{2n}$. Proof (induction) base cases: $n=1$:

$x^2-y^2 = (x+y)(x-y)$ ✓ $n=2$: $x^4-y^4 = (x^2-y^2)(x^2+y^2) = (x-y)(x+y)(x^2+y^2)$ ✓ □.

Induction step: consider $(x^{2n}-y^{2n})(x^2+y^2) = x^{2n+2}-y^{2n+2}+x^{2n}y^2 - x^2y^{2n}$

This implies that: $x^{2n+2}-y^{2n+2} = (x^{2n}-y^{2n})(x^2+y^2) - x^2y^2(x^{2n-2}-y^{2n-2})$

so assuming general induction hypothesis $p(n), p(n-1) \Rightarrow p(n)$, so result holds by induction. □

Q16 Thm $\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n} < 1$ Proof Set $s_n = \frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n}$. We will show: $s_n \leq 1-\frac{1}{n+1}$ Proof (by induction) base case: $n=1$ $s_n = \frac{1}{2} \leq 1-\frac{1}{2}$ ✓ □.

induction step: note: $s_{n+1} = s_n - \frac{1}{n+1}+\frac{1}{2n+1}+\frac{1}{2n+2} = s_n + \frac{1}{2n+1}-\frac{1}{2n+2} = s_n + \frac{2n+2-(2n+1)}{(2n+1)(2n+2)}$

$= s_n + \frac{1}{(2n+1)(2n+2)}$ By induction hypothesis for n, this implies $s_{n+1} \leq 1 - \frac{1}{n+1} + \frac{1}{(2n+1)(2n+2)}$

note: $\frac{1}{(2n+1)(2n+2)} \leq \frac{1}{(n+1)(n+2)} = \frac{1}{n+1}-\frac{1}{n+2} \Rightarrow -\frac{1}{n+1}+\frac{1}{(2n+1)(2n+2)} \leq -\frac{1}{n+2}$ (*)

so (*) $\Rightarrow s_{n+1} \leq 1-\frac{1}{n+2}$, so $p(n) \Rightarrow p(n+1)$ as required □.

Q17 Thm $1-\frac{1}{2}+\frac{1}{3}-\dots-\frac{1}{2n} = \frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n}$ Proof (by induction) base case $n=1$:

$1-\frac{1}{2}=\frac{1}{2}$, $\frac{1}{2}=\frac{1}{2}$ ✓ □. Induction step: set $l_n = 1-\frac{1}{2}+\frac{1}{3}-\dots-\frac{1}{2n}$ $r_n = \frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n}$.

induction hypothesis, assume $l_n = r_n$ for some n. Consider $l_{n+1} = l_n + \frac{1}{2n+1}-\frac{1}{2n+2}$

$\Rightarrow l_{n+1} = l_n + \frac{1}{(2n+1)(2n+2)}$. Now consider $r_{n+1} = r_n - \frac{1}{n+1}+\frac{1}{2n+1}+\frac{1}{2n+2} = r_n + \frac{1}{2n+1}-\frac{1}{2n+2} = r_n + \frac{1}{(2n+1)(2n+2)}$

so $r_{n+1} = r_n + \frac{1}{(2n+1)(2n+2)} \underset{\uparrow \text{ uses } p(n)}{=} l_n + \frac{1}{(2n+1)(2n+2)} = l_{n+1}$, as required $\square$.

Q18 Not equivalence relation as not transitive: $6 \sim \frac{1}{2}$, $6 \sim \frac{1}{3}$ by $\frac{1}{2} \not\sim \frac{1}{3}$

Q19 ~~Yes~~ No: reflexive: choose $A=1, B=0$, then $f(x) \le f(x) \Rightarrow f \sim f$.

not symmetric: note $0 \sim x$, as choose $A=0, B=0 \Rightarrow 0 \le 0$ ✓.

but $x \not\sim 0$ as $x \le A.0 + B \Rightarrow x \le B$ but this does not hold for $x \ge B+1$.

Q20 ~~Yes~~ No: reflexive: choose identity map $f: A \to A$, bijection, so $A \sim A$.

not symmetric $A = \{1,2\}$, $B = \{3\}$, then $\exists f: A \to B$ surjective, also $A \sim B$, but no map $f: B \to A$ surjective, so $B \not\sim A$.

Q21 No: not symmetric: $f = x$, $g = 1$, then $f \sim g$ but $g \not\sim f$ as $(1)' = 0 \not\sim x$.