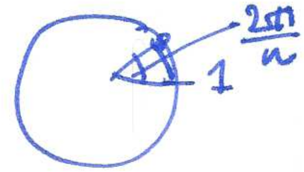
General case for regular polygons:

symmetries generated by rotation $z \mapsto e^{\frac{2\pi}{n}i}z$

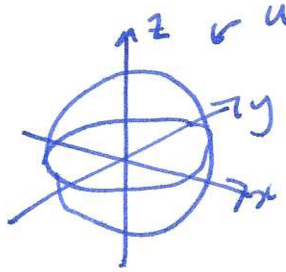
and reflection $z \mapsto \bar{z}$



so group has order $2n$, elements $\{1, e^{\frac{2\pi}{n}i}z, e^{\frac{2\pi}{n}i\cdot 2}z, \ldots e^{\frac{2\pi}{n}i(n-1)}z,$
$\bar{z}, e^{\frac{2\pi}{n}i}\bar{z}, \ldots \}$

Spherical geometry

S^2 unit sphere $x^2+y^2+z^2=1$.



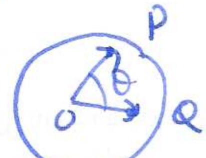
point $\Leftrightarrow$ unit vector

antipodal point $\Leftrightarrow$ negative of vector.

line $\Leftrightarrow$ intersection of plane through O with S^2

planes described by $\perp$ vector $\underline{n}$, i.e. $\underline{n}\cdot\underline{x}=0$.

· How to find spherical distance:



recall: $\overline{OP}\cdot\overline{OQ} = \|OP\|\|OQ\|.\cos\theta$
$= \cos\theta$

distance = angle in radians!

so $d_{S^2}(P,Q) = \cos^{-1}(\overline{OP}\cdot\overline{OQ})$

Example find distance between $P=(-\frac{1}{\sqrt2}, \frac{1}{\sqrt2}, 0)$ and $Q=(\frac{1}{\sqrt3}, -\frac{1}{\sqrt3}, \frac{1}{\sqrt3})$.

· How to find a line containing two points, ~~say P~~

$\overline{OP}\times\overline{OQ}$ $\perp$ to both vectors, so normal vector $\underline{n}$ to plane.

Example find circle containing $P=(\frac{3}{5}, -\frac{4}{5}, 0)$ $Q=(0, \frac{\sqrt3}{2}, -\frac12)$.

$$\vec{OP}\times\vec{OQ} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3/5 & -4/5 & 0 \\ 0 & \frac{\sqrt3}{2} & -\frac12 \end{vmatrix} = \langle \frac{2}{5}, \frac{3}{10}, \frac{3\sqrt3}{10} \rangle \leftarrow \text{not unit vector!}$$

parallel to $\langle 4, 3, 3\sqrt3\rangle \cdot \frac{1}{\sqrt{52}}$

so equation of plane is $4x+3y+3\sqrt3 z = 0$.

· How to find the angle between two lines

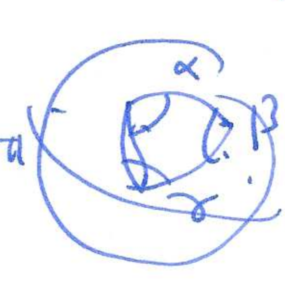
angle same as angle between normal vectors!

Example L_1 has $\underline{n}_1 = \frac13\langle 1,2,-2\rangle$ L_2 has $\underline{n}_2 = \frac14\langle 1,-3,\sqrt6\rangle$

$\theta = \cos^{-1}(\underline{u}_1 \cdot \underline{u}_2)$ ← if unit vectors! otherwise need $\theta = \cos^{-1}\left(\frac{\underline{u}_1 \cdot \underline{u}_2}{\|\underline{u}_1\| \, \|\underline{u}_2\|}\right)$.

- How to find the area of a triangle

For a spherical triangle PQR, area $= \alpha + \beta + \gamma - \pi$



– find lines
– find angles between lines
– use Gauss/Girard's thm.

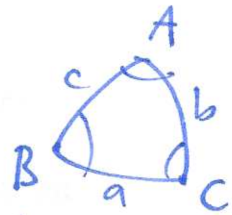
Example ① $P = (-1,0,0)$, $Q = (0,-1,0)$, $R = (0,0,1)$

② $P = \left(\frac{3}{5}, -\frac{4}{5}, 0\right)$, $Q = \left(0, -\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ $R = \left(\frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}\right)$

③ $P = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$, $Q = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$, $R = (1,0,0)$.

Spherical trigonometry

cosine rule



(*) $\sin a \sin b \cos C = \cos c - \cos a \cos b$

special case (spherical pythagorean thm, $C = \frac{\pi}{2}$) : $\cos c = \cos a \cos b$

~~Proof (of (*)) $A \cdot B = \cos c$, $A \cdot C = \cos b$, $B \cdot C = \cos a$.~~

sin rule : $\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$

Fact congruence for spherical triangles follows from: SSS → cosine law gives angle

SAS, ASA → cosine law AAA   AAA ⇒ SSS.

cosine law SSS

Spherical isometries ↔ rotations of $\mathbb{R}^3$ about O!.

examples $f_1(x,y,z) \mapsto (-x,y,z)$ reflection in ~~equ~~ yz-plane

$f_2(x,y,z) \mapsto (x,-y,z)$ reflection in xz-plane

$f_3(x,y,z) \mapsto (x,y,-z)$ reflection in xy-plane (equator)

$f_1 \circ f_2\, (x,y,z) \mapsto (-x,-y,z)$ ← rotation about z-axis by π radians.