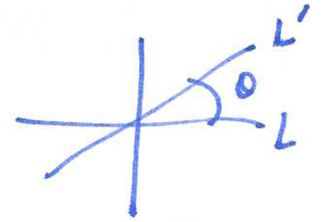


__case 2__ wlog $L =$ x-axis, $L \cap L' = 0$, angle $= \theta$

$\rho_L : z \mapsto \bar{z}$ $\quad \rho_{L'} : z \mapsto e^{-i\theta} z \mapsto e^{i\theta} \bar{z} \mapsto e^{2i\theta} \bar{z}$

$\rho_{L'} \rho_L = \quad z \mapsto \bar{z} \mapsto e^{2i\theta} z$, rotation by 2θ. $\square$.

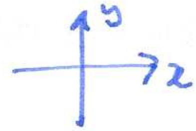
Q: are there any other isometries. A: yes.

__§6.3 Glide reflections__

__Defn__ given two points A, B, $\rho_{AB} \circ T_{AB}$ is called the __glide reflection__ Γ_{AB}.

__Observation__ reflections are a special case of glide reflections.

In coordinates: special case $L = x$-axis $\quad z \mapsto \bar{z} + c$

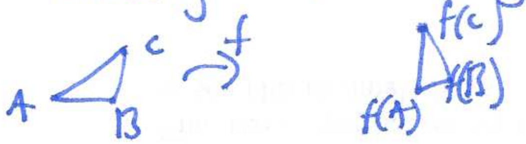


general case: conjugate with map $L' \to L$.

Q: are there any other isometries?

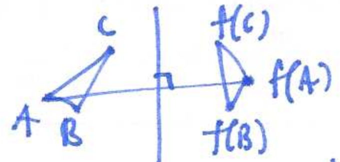
__§6.4. Classification of isometries__.

__recall__ f isometry determined by image of 3 non-colinear points (i.e. a triangle).



__Theorem__ Every isometry is the product of at most 3 reflections

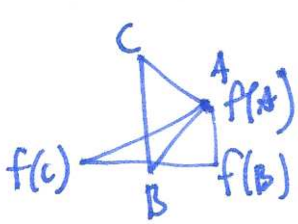
__Proof__ Let ABC be a triangle, and consider $f(A)f(B)f(C)$ ← congruent triangle. __case__: no vertices in common: $f(A) \neq A$ etc. construct $\perp$ bisector to $Af(A)$ and reflect in that line.



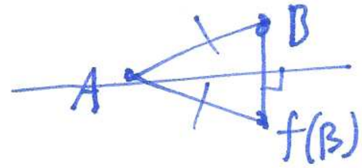
takes $f(A)$ to A.

so after at most 1 reflection, ABC, $f(ABC)$ share a vertex.

__Case__ 1 vertex in common (wlog A up to relabelling).

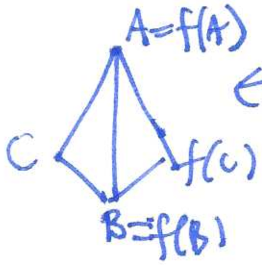


aim fix $A=f(A)$, move $f(B)$ to B.



isosceles triangle, so $\perp$ bisector passes through A, so reflection in $\perp$ bisector fixes $A=f(A)$ and sends $f(B)$ to B.

case 2 2 vertices in common, wlog $A=f(A)$, $B=f(B)$.



$\leftarrow$ reflect in AB.

$\square$.

Theorem Every isometry is either a translation, rotation, reflection or glide reflection.

Proof Every isometry product ~~of~~ of at most 3 reflections.

0 reflections : id.

1 reflection : reflection

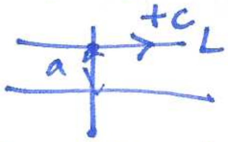
2 reflections : parallel lines $\rightsquigarrow$ translation
not parallel $\rightsquigarrow$ rotation.

3 reflections : claim : this is a glide reflection.

Proof (coordinates) ① wlog reflection is $z \mapsto \bar{z}$ and translation is $z \mapsto z+w$.

so composition is $z \mapsto \bar{z}+w$.

observation :



L is $z \mapsto z-ai \mapsto \bar{z}+ai+c \mapsto \bar{z}+\underbrace{2ai+c}_{w}$

so $z \mapsto \bar{z}+w$ is glide reflection in $y=\frac{1}{2}\operatorname{Im}(w)$ w/ translation part $+c=\operatorname{Re}(w)$

② $z \mapsto \bar{z}$, $z \mapsto e^{i\theta}z+w$, composition : $z \mapsto e^{i\theta}\bar{z}+w$. $\Leftarrow$ conjugate this by $z \mapsto e^{i\theta/2}z$, get $z \mapsto \bar{z}+w$, glide reflection parallel to x-axis, so $z \mapsto e^{i\theta}\bar{z}+w$ is glide reflection in line at angle $\theta/2$. $\square$.

Corollary Any ~~rotat~~ isometry can be written as $z \mapsto e^{i\theta}z+w$
or $z \mapsto e^{i\theta}\bar{z}+w$. $\square$.

Corollary Any map of that form is an isometry. $\square$.

Def^{n} ① an isometry is <u>orientation preserving</u> if it is a composition of an even number of reflections.

② order points in triangle ABC  (anticlockwise) and look at order $f(A)f(B)f(C)$ ← same (anticlockwise), isometry is orientation preserving
← different (clockwise) isometry is orientation reversing

Fact: translations, rotations orientation preserving. ← index 2 subgp!
reflections, glide reflections orientation reversing.

Fact

	+	−
+	+	−
−	−	+

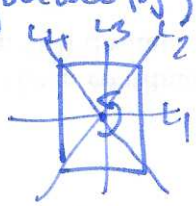
Fact

	orientation preserving +	reversing −
fixed points	rotation	reflection
no fixed points	translation	glide reflection.

§6.5 symmetries of polygons

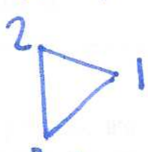
Def^{n} A <u>symmetry</u> of a polygon Υ is an isometry s.t. $f(\Upsilon) = \Upsilon$.

(id is always a symmetry).

Example square:  rotations by 0, 90, 180, 270.
reflections. ρ_{L_i}

note collection of symmetries: contains id.
if f, g are symmetries so if f^{-1}, $f \circ g$

so symmetries of Υ is a <u>group</u> (subgp of all isometries).

Example equilateral triangle  **note** symmetry group = symmetric gp in 3 elements.

note $\mathrm{sym}(\text{square}) \neq S_4$!