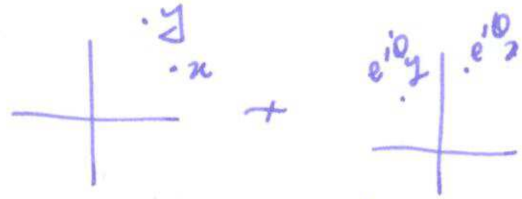
<u>coordinates</u> <u>special rotation</u> about O : $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

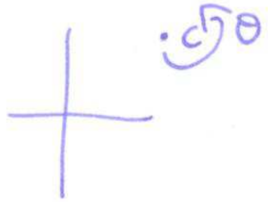
$z \mapsto e^{i\theta} z$

check isometry:



$$d(e^{i\theta}x, e^{i\theta}y) = |e^{i\theta}x - e^{i\theta}y| = |e^{i\theta}||x-y| = |x-y|$$

<u>general rotation</u> :



$$R_{c,\theta} = \tau_c^{+1} R_{0,\theta} \tau_c^{-1}$$

$z \mapsto z+c$ $\quad z \mapsto e^{i\theta}z$ $\quad z \mapsto z-c$

$$z \mapsto e^{i\theta}(z-c)+c = e^{i\theta}z + c(1-e^{i\theta})$$

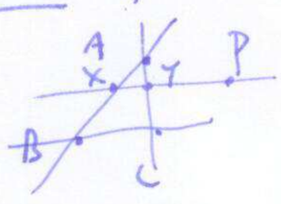
<u>Useful facts</u> <u>Propn</u> isometries take straight lines to straight lines

<u>Propn</u> isometries take circles to circles.

<u>Propn</u> If A, B, C are three non-colinear points, and f, g are isometries with $f(A) = g(A)$, $f(B) = g(B)$, $f(C) = g(C)$, then $f = g$.

<u>Proof</u> Note that $f^{-1}g$ fixes A, B, C 3-non-colinear points.

<u>claim</u> if an isometry fixes 3 non-colinear points, it is the identity



fix $A, B \Rightarrow$ ~~fix~~ preserve straight line through A, B.

$\Rightarrow$ fix straight line through A, B pointwise.

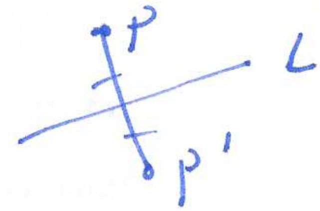
similarly for BC, AC.

for any point P, draw line intersecting two of the the straight lines at different points. X, Y fixed $\Rightarrow \overline{XY}$ fixed $\Rightarrow P$ fixed $\square$.

so $f^{-1}g = id \Rightarrow g = f$ $\square$.

§6.2 Reflections

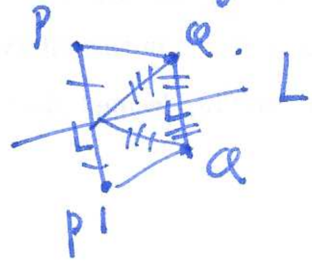
Defn given a straight line L and a point P, construct $\perp$ to L through P, and choose P' s.t. dist P to L = dist P' to L.



Call this ρ_L

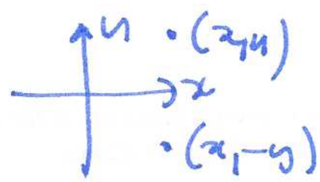
Proposition ρ_L is an isometry

Proof (geometric)



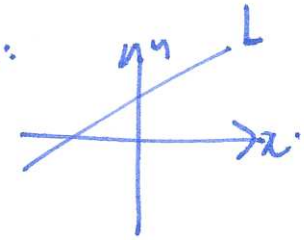
~~similar~~ cong triangle $\Rightarrow |PQ| = |P'Q'|$ $\square$

In coordinates: favourite reflection



$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$ or $z \stackrel{K}{\longmapsto} \bar{z}$.

Any reflection:



let τ be a map taking L to x-axis

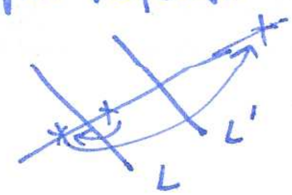
then $\tau^{-1} K \tau$ is a reflection in L.

claim for any L there is an isometry τ taking L to x-axis.

observation if ρ_L is a reflection, $\rho_L^2 = id \Rightarrow \rho_L = \rho_L^{-1}$.

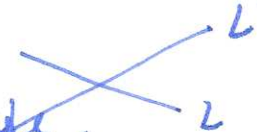
Q: what is the composition of two reflections?

A: $\rho_L \rho_{L'}$ case $L \parallel L'$:



translation by twice $\perp$ vector from L to L'.

case $L \nparallel L'$ then they intersect:

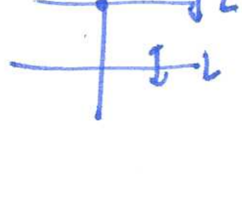


note: $\tau_L \tau_{L'}$ fixes intersection pt!

claim $\rho_L \rho_{L'}$ = rotation by twice angle.

Proofs (in coordinates) case 1 wlog L = x-axis, L' parallel above,

so L' is $y = c$



$\rho_L : z \mapsto \bar{z}$

$\rho_{L'} : z \mapsto z - ic \mapsto \bar{z} + ic \mapsto \bar{z} + 2ic$