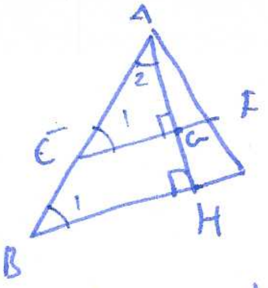
construct perpendicular:
$\triangle ABH$ similar to $\triangle AEG$
by ~~SAS~~ AAA



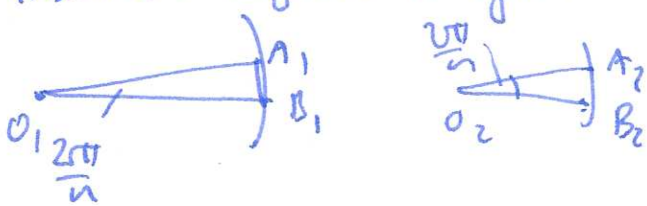
$\Rightarrow \frac{|AE|}{|AB|} = \frac{|AG|}{|AH|}$

area $= \frac{1}{2}$ base $\times$ height, so $|\triangle ABC| = \frac{1}{2}|BC|\,|AH|$.

$|\triangle AEF| = \frac{1}{2}|EF|\,|AG|$.

so $\frac{|\triangle ABC|}{|\triangle AEF|} = \frac{|BC|\,|AH|}{|EF|\,|AG|} = \frac{|AB|}{|AE|}\cdot\frac{|AB|}{|AE|} = \frac{|AB|^2}{|AE|^2}$ $\square$.

reasonable assumption: for large n, area of circle $\sim$ area of inscribed regular n-gon.



$\frac{A_1}{A_2} = \frac{n\,\triangle O_1 A_1 B_1}{n\,\triangle O_2 A_2 B_2} = \frac{r_1^2}{r_2^2}$ $\square$.

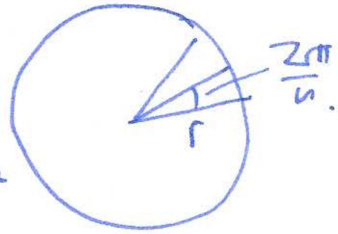
alternate proof: use calculus. $\square$.

<u>Corollary</u> there is a constant π s.t. $A = \pi r^2$.

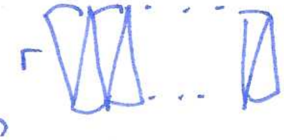
<u>(4.4.5)</u> The proportionality constants for circumference and area of circles are related as $\alpha = 2\pi$.

<u>Proof</u> (sketch)

circumference $C = \alpha r$
area $A = \pi r^2$



$\sim$



approx rectangle $r \times \frac{C}{2} = A$.

$r\frac{\alpha r}{2} = \pi r^2$

$\Rightarrow \alpha = 2\pi$. $\square$.

Q: which regular polygons can be constructed?

Euclid: $n = 3, 4, 5, 6$, can bisect angles, so this gives $2^a \cdot n$ for those n.

[Gauss 1796] $n = 17$, in fact can construct regular n-gon for primes s.t. $p = 2^{2^n}+1$.

Thm Can construct regular n-gon iff $n = 2^k p_1 p_2 \cdots p_m$ where each p_i is a prime of the form $2^{2^n}+1$.

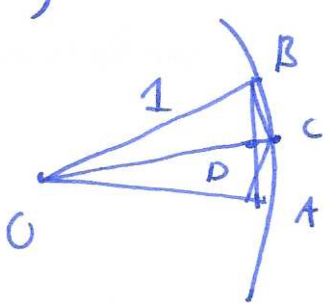
Q: which primes have this form?

$$2^{2^0}+1 = 2^1+1 = 3$$
$$2^{2^1}+1 = 2^2+1 = 5$$
$$2^{2^2}+1 = 2^4+1 = 17$$
$$2^{2^3}+1 = 2^8+1 = 257$$
$$2^{2^4}+1 = 2^{16}+1 = 65,537.$$

[Gauss 1796] $\Leftarrow$
[Wantzel 1837] $\Rightarrow$.

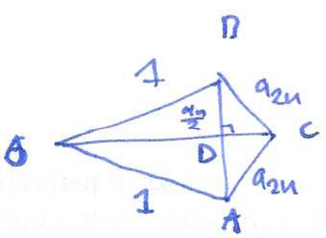
not known if there are any others! (don't know if $2^{2^{20}}+1$ is prime!).

Approx of π (first approx by Archimedes)



$a_n = |\text{chord } AB|$
$\quad =$ side of regular n-gon.

$C =$ midpoint of $\overset{\frown}{AB}$



$a_{2n} = |\text{chord } AC|$

ΔADC, ΔOAD right angled, so $a_{2n}^2 = |AC|^2 = |AD|^2 + |DC|^2$

$$a_{2n}^2 = \left(\frac{a_n}{2}\right)^2 + \left(|OC| - |OD|\right)^2$$

$$a_{2n}^2 = \frac{1}{4}a_n^2 + \left(1 - \sqrt{|OA|^2 - |AD|^2}\right)^2$$

$$= \frac{1}{4}a_n^2 + \left(1 - \sqrt{1 - \left(\frac{a_n}{2}\right)^2}\right)^2$$

$$a_{2n}^2 = \cancel{\frac{1}{4}a_n^2} + 1 - 2\sqrt{1-\left(\frac{a_n}{2}\right)^2} + 1 - \cancel{\left(\frac{a_n}{2}\right)^2} = 2 - \sqrt{4 - a_n^2}$$

Prop 4.4.7 Length a_n of side of regular polygon in circle of radius 1 satisfies $a_{2n} = \sqrt{2 - \sqrt{4 - a_n^2}}$.

recall: reg hexagon has side length 1, so $a_6 = 1$

$a_{12} = \sqrt{2-\sqrt{3}} \simeq 0.51763\cdots$

$a_{24} = \sqrt{2-\sqrt{2+\sqrt{3}}} \simeq 0.261052\cdots$

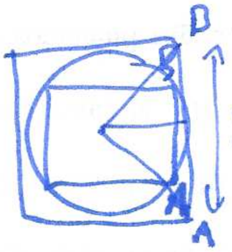
$a_{48} = \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}} \simeq 0.1308062$

$a_{96} \approx \qquad \simeq 0.06543816\cdots$

length of arc > length of chord, $2\pi > n a_n$, $\pi > \frac{n}{2} a_n$

e.g. $\pi > 48 a_{96} = 3.14103195089\ldots$

upper bounds:



consider smallest n-gon containing circle.

$b_n > \overset{\frown}{AB}$.

claim $b_6 = \frac{2}{\sqrt{3}}$

$$b_{2n} = \frac{2\left(\sqrt{4+b_n^2}-2\right)}{b_n} \quad \text{or} \quad b_n = \frac{2a_n}{\sqrt{4-a_n^2}}$$

Fact $b_{96} = 0.654732208\cdots$ so $\pi < \frac{96\, b_{96}}{2} = 3.1427145\cdots$

Prop 4.4.8 (Archimedes)

$$3\frac{10}{71} < \pi < 3\frac{1}{7} \quad \left(= \frac{22}{7}\right)$$

Thm (von Lindemann 1882) Given line segment $|AB| = a$. Impossible to construct a square whose area is equal to that of a given circle of radius $|AB| = a$