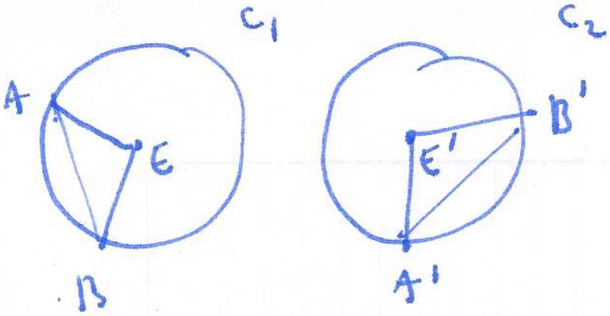



want to show TFAE

1) $\widehat{AB} = \widehat{A'B'}$

2) $|AB| = |A'B'|$

3) $\angle AEB = \angle A'E'B'$

<u>Proof</u> 1) $\Rightarrow$ 2) assume $\widehat{AB} = \widehat{A'B'}$ Apply/translate circle $C_1$ to $C_2$ s.t. $AE$ falls on $A'E'$ and $\widehat{AB}$ on $\widehat{A'B'}$

as $\widehat{AB} = \widehat{A'B'} \Rightarrow$ $B$ falls on $B'$.

$\Rightarrow$ chord $AB$ falls on chord $A'B'$

$\Rightarrow |AB| = |A'B'|$

2) $\Rightarrow$ 3) Assume $|AB| = |A'B'|$, so $\triangle AEB \cong \triangle A'E'B'$

$\Rightarrow \angle AEB = \angle A'E'B'$

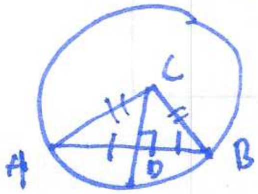
3) $\Rightarrow$ 2) Assume $\angle AEB = \angle A'E'B'$ Apply $C_1$ to $C_2$ s.t. $E$ lies on $E'$

angles coincide, equal radius $\Rightarrow$ $A$ lies on $A'$, $B$ lies on $B'$

$\Rightarrow \widehat{AB} = \widehat{A'B'}$ $\square$.

<u>Corollary</u> B1 Def$^n$ 17 In a circle all semicircles are equal to one another. $\square$.

<u>(4.1.3) B3P3</u> In a circle, a radius bisects a chord not through the center iff the radius and chord are $\perp$



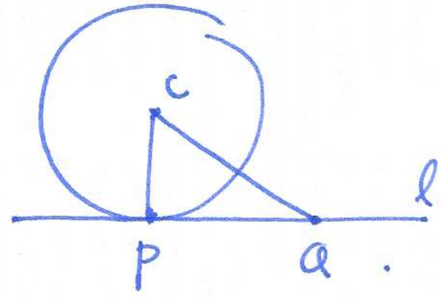

<u>Proof</u> Suppose $D$ is midpoint $\Rightarrow$ cong triangles

$\Rightarrow \angle ADC = \angle BDC = \pi/2$

suppose $CD \perp AB$, $ACB$ isosceles $\Rightarrow$ $\angle B = \angle A$ other $90^\circ$ $\Rightarrow \angle ACD = \angle DCB$ $\Rightarrow \triangle$ cong $\square$.

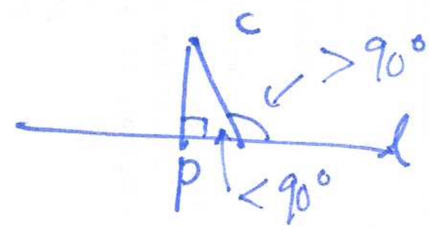
Defn: A straight line is tangent to a circle if it intersects in one point.

(4.1.4) B3 P16, P18 If a straight line intersects a circle, it is tangent iff line is $\perp$ to the radius at the intersection point.




Proof $\Rightarrow$ assume tangent:

claim: P closest point on line to C, as any other point Q on l lies outside circle.




claim: closest point lies on $\perp$:
follows from greater angle subtends greater side $\square$.

$\Leftarrow$ assume $CP \perp$ to $l$ $\Rightarrow$ ~~C~~P is closest point on $l$ to C (follows from greater side subtends greater angle!)
$\Rightarrow$ every other point on $l$ lies outside circle $\Rightarrow$ tangent. $\square$

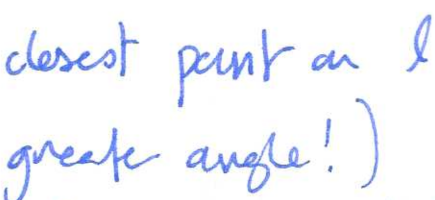
(4.1.5) B6 P33 In equal circles, central angles are proportional to arcs they subtend.

Proof only works for rational ratios! $\square$.

Eratosthenes ~ 200BC Alexandria sun makes angle of $1/50$ of $360°$ ($7.2°$)

5000 stadia

$\Leftarrow$ Syene sun shines straight down well in summer solstice

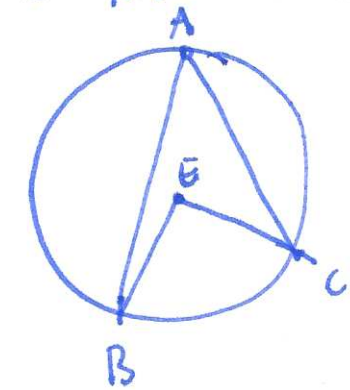



$$\frac{\text{circ of earth}}{5000 \text{ stadia}} = \frac{360°}{7.2°} = 50$$

$\Rightarrow$ 250,000 stadia = 40,000 km, accurate to 10%.

## §4.2 The non-neutral (Euclidean) geometry of the circle

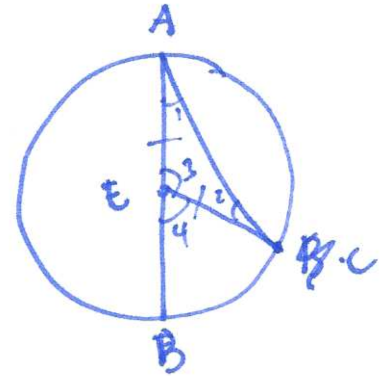
(4.2.1) B3 P20 The angle at the center of the circle is twice the angle subtended at the circumference.



$\angle BEC = 2\angle BAC$

**Proof** special case:

AB goes through E



$\triangle ACE$ isoceles.

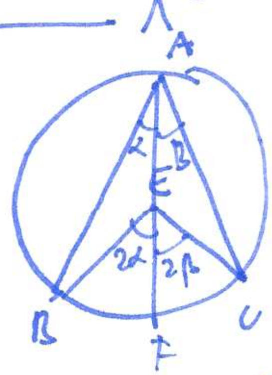
so $\angle 1 = \angle 2$

$2\angle 1 + \angle 3 = 180^\circ$

$\angle 3 + \angle 4 = 180^\circ$

$\Rightarrow \angle 4 = 2\angle 1$ $\square$.

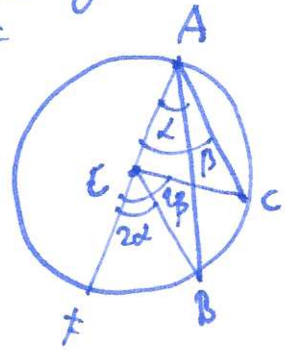
case 1 center E lies in interior of ABC:



by special case $\angle BAC = \alpha + \beta$

$\angle BEC = 2\alpha + 2\beta$.

case 2

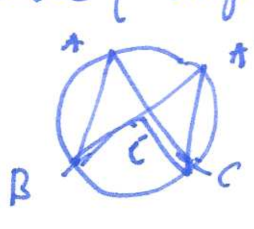


$\angle CAB = \beta - \alpha$

$\angle CEB = 2\beta - 2\alpha$ $\square$.

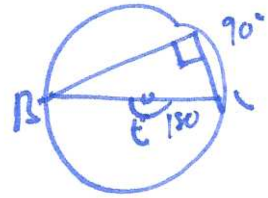
(4.2.2) B3 P21 (corollary) In a circle, angles subtended by a common segment are equal.

**Proof**



$\square$.

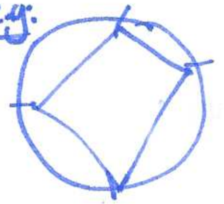
**(4.2.3) BP31** In a circle, the angle subtended by a diameter is 90°. **Proof**




□.

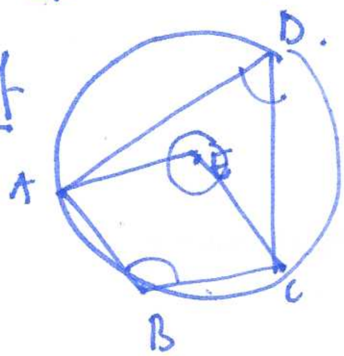
**Defn** A polygon is **cyclic** if all of its vertices lie on a circle.

e.g.

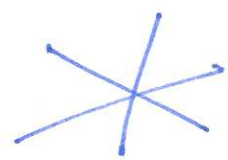


**(4.2.6) B3P22** The opposite angles of a cyclic quadrilateral sum to 180°.

**Proof**



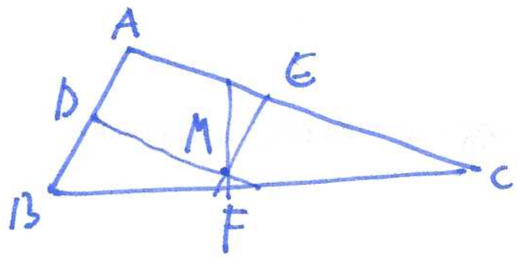

angles at E twice as big as angles at D and B. □.

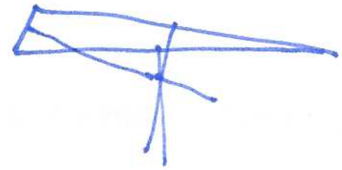


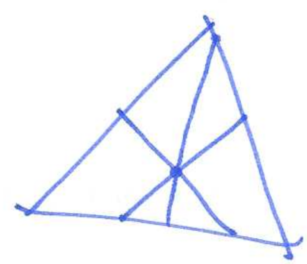
**Defn** 3 lines are concurrent if they contain a common point

**Prop 4.2.7** The perpendicular bisectors of the sides of a triangle are concurrent

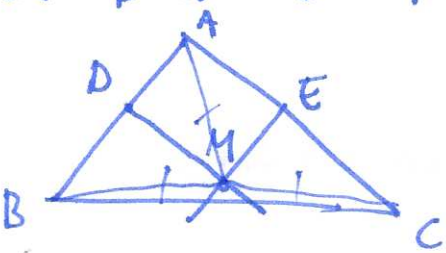
**Proof**





Let D and E intersect at M




recall: $\perp$ bisector = set of equidistant points

$$\left.\begin{aligned} |AM| &= |BM| \\ |AM| &= |CM| \end{aligned}\right\} \quad |BM| = |CM|$$

$\Rightarrow$ M lies on $\perp$ bisector at F.

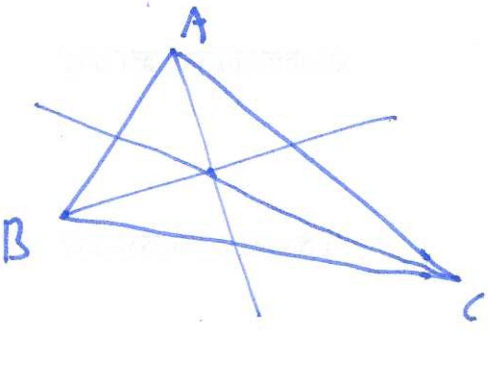
□.

(4.2.8) B4 P5 Circumscribe a circle about a given triangle. (24)

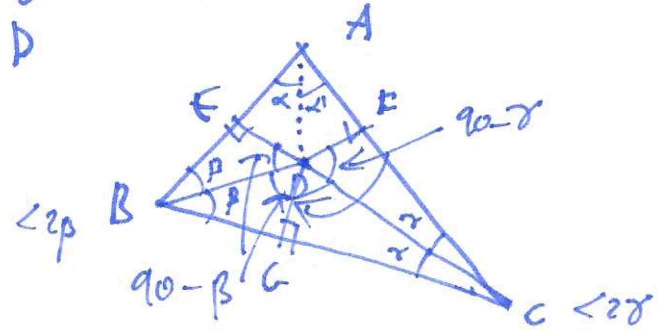
(corollary! of above) □

(4.2.9) The angle bisectors of ~~the~~ a triangle are concurrent.

Proof




Let angle bisectors at B and C meet at D




Construct perpendiculars through D
then angles around D are $90-\beta$, $90-\beta$, $90-\gamma$, $90-\gamma$
so remaining angle is $2\beta+2\gamma$
note: $\triangle BDE \cong \triangle BDG$ and $\triangle CDE \cong CDG$
so $|DE| = |DG| = |DF|$, note: $90°$ triangle and SS
$\Rightarrow$ SSS $\Rightarrow$ $\triangle ADE \cong \triangle ADF$ $\Rightarrow$ $\alpha = \alpha'$
so D lies on angle bisector through A. □.

(4.2.10) B4 P4 Can inscribe a circle inside a given triangle. Pf Corollary of above □.