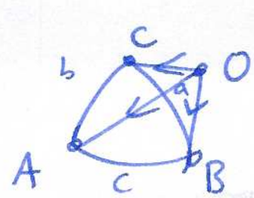
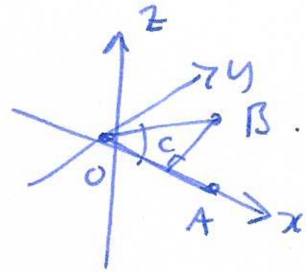
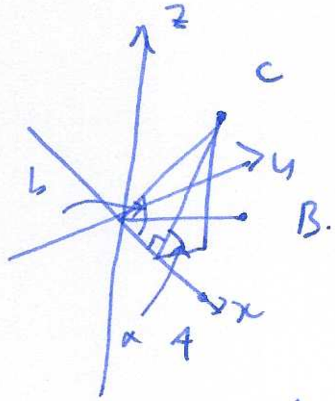
Proof (of spherical sine rule) ($R=1$)



can choose $\underline{OA} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$



$\underline{OB} = \begin{bmatrix} \cos c \\ \sin c \\ 0 \end{bmatrix}$



$\underline{OC} = \begin{bmatrix} \cos b \\ \sin b \cos\alpha \\ \sin b \sin\alpha \end{bmatrix}$

vol of parallelepiped = triple cross product = $\det\begin{bmatrix} OA & OB & OC \end{bmatrix}$

$\underline{OA}\cdot(\underline{OB}\times\underline{OC})$

$$\begin{vmatrix} 1 & \cos c & \cos b \\ 0 & \sin c & \sin b\cos\alpha \\ 0 & 0 & \sin b \sin\alpha \end{vmatrix} = \sin c \sin b \sin\alpha = V$$

but could put OB on x-axis, then OC in xy plane, gives

$$V = \sin a \sin c \sin\beta$$

so $V = \sin c \sin b \sin\alpha = \sin a \sin c \sin\beta \Rightarrow \dfrac{\sin\alpha}{\sin a} = \dfrac{\sin\beta}{\sin b}$ $\square$

Fact spherical area determined by angles:

(radius R) area of triangle $= (\alpha+\beta+\gamma-\pi)R^2$.

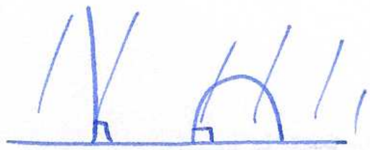
Example

$\alpha=\beta=\gamma=\frac{\pi}{2}$

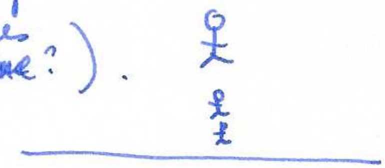
$$\text{Area} = \left(\frac{\pi}{2}+\frac{\pi}{2}+\frac{\pi}{2}-\pi\right)R^2 = \frac{\pi}{2}R^2$$

area of sphere $= 4\pi R^2$, triangle is $1/8$ this area.

Hyperbolic geometry:  less familiar, here's a specific model:

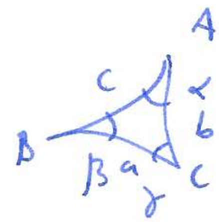
upper half space model:  $\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$.

- points ↔ points
- straight lines ↔ vertical straight lines, circles ⊥ to x-axis
  ↑ infinitely many "parallel" lines (or none?).
- angles ↔ same as in $\mathbb{R}^2$
- metric: shrinks as you go towards x-axis.



There are analogues of sine + cosine rules:

sine: $$\frac{\sin \alpha}{\sinh a} = \frac{\sin \beta}{\sinh b} = \frac{\sin \gamma}{\sinh c}$$



cosine: $$\cosh a = \cosh b \cosh c - \cos \alpha \sinh b \sinh c$$
$$\cos \alpha = \cosh a \sin \beta \sin \gamma - \cos \beta \cos \gamma$$

area determined by angles: $\text{area} = \pi - \alpha - \beta - \gamma$

Euclid: innovation: assume certain axioms, make deductions from them.

Neutral geometry

1st 4 postulates → spherical ← no parallel line

→ euclidean ← unique parallel line

→ hyperbolic ← ∞-many parallel lines.

Q: what is the definition of a polygon?

triangle  quad  what about: