**Math 623 Geometry Fall 22 Sample Final**

(1) Prove that if the edges of a quadrilateral are all the same length, then it is a parallelogram.

(2) Show that given two points $A$ and $B$, the set of equidistant points from both $A$ and $B$ is the perpendicular bisector to the line segment $\overline{AB}$.

(3) Given two lines $L$ and $M$, show that the points on the lie forming the angle bisectors are equidistant from both $L$ and $M$.

(4) Show that every polygon may be divided into triangles by adding edges connecting vertices. Use this to compute the sum of the interior angles of an $n$-gon.

(5) Prove the following:
   (a) Let $m$ and $n$ be two parallel lines and let $AB$ be a directed line segment that first intersects $m$ and then $n$ and whose length is twice the distance between $m$ and $n$. Then prove that $\rho_n \circ \rho_m = \tau_{AB}$.
   (b) Let $m$ and $n$ be two straight lines that intersect at a point $A$ and let $\alpha$ be the counterclockwise angle from $m$ to $n$ at $A$. Then prove that $\rho_n \circ \rho_m = R_{A,2\alpha}$.

(6) Construct a spherical triangle with three right angles. What is its area? Is there a spherical quadrilateral with four right angles?

(7) Euler's formula for a torus (i.e. hollowed out donut) is $v - e + f = 0$. If a torus has exactly 20 triangular faces, how many vertices and edges does it have ?

(8) Does there exist a polyhedron in $\mathbb{R}^3$ consisting of 12 hexagonal faces such that every vertex has degree 4? Please justify your answer.

(9) Define the antipodal map and show that it is a spherical isometry.

(10) (a) In the Projective plane define the following:
   (i) ideal point (ii) projective line (iii) ideal line

   (b) Prove that every two distinct projective lines intersect.

(11) For the three statements below, fill in the chart with **A**(Always) or **S** (Sometimes) or **N**(Never) in each space.

$X_1$: Any two distinct lines intersect in a unique point.
$X_2$: For a line $L$, and point $Q$ outside $L$, there is a unique line is parallel to $L$ through $Q$.
$X_3$: The sum of angles of a triangle are greater than $\pi$.

| | $X_1$ | $X_2$ | $X_3$ |
|---|---|---|---|
| $\mathbb{R}^2$ | | | |
| $S^2$ | | | |
| $\mathbb{H}^2$ | | | |

(12) Find angle between the great circles $L_{\langle 1/3,2/3,2/3\rangle}$ and $L_{\langle -3/5,4/5,0\rangle}$.

(13) Prove that any two distinct non-antipodal points on $S^2$ lie on a unique great circle.

(14) Find the great circle containing the points $P = (0, 1/2, \sqrt{3}/2)$ and $Q = (-3/5, 0, 4/5)$.

(15) Let $f$ be any isometry of $\mathbb{R}^2$. Show that $f$ preserves angles.

(16) Show that the composition of a rotation by $\pi$ about $(0, 0)$ followed by a rotation by $\pi$ about $(2, 0)$ is a translation. Draw pictures.

(17) Let $f$ be an isometry of $\mathbb{R}^2$ given by three reflections, $f = \rho_c \circ \rho_b \circ \rho_a$. Suppose the three lines $a$, $b$ and $c$ intersect at a single point. Describe the isometry $f$.

(18) Let $f$ be any isometry of $\mathbb{R}^2$. Show that $f$ takes circles to circles.

(19) If a point lies on a great circle, then its antipode also lies on it i.e if $P \in L_{\mathbf{n}}$ then $-P \in L_{\mathbf{n}}$.

(20) Any two distinct great circles intersect in a pair of antipodal points i.e. if $\mathbf{n}_1 \neq \mathbf{n}_2$ then $L_{\mathbf{n}_1} \cap L_{\mathbf{n}_2} = \{P, -P\}$.

(21) Let $P$ and $Q$ be distinct points on $S^2$. Then the set of points which are equidistant from $P$ and $Q$ is a great circle.

(22) Let $f$ be an isometry of $R^2$ given by three reflections, $f = \rho_c \circ \rho_b \circ \rho_a$.
  (a) Suppose the three lines intersect at a single point. Describe the isometry $f$.
  (b) Suppose that $a$ is parallel to $b$, and $c$ is perpendicular to $a$. Describe the isometry $f$.

(23) Let $f$ be reflection in the line $x = 1$, and let $g$ be an anti-clockwise rotation of $\pi/4$ about $(0, 0)$. Describe $f \circ g$.

(24) Give example of two different type of isometries which preserve the $x$-axis.

(25) Describe all isometries which preserve the $x$-axis.

(26) Find a translation of $\mathbb{R}^2$ which takes the line $x = y$ to the lines $x = y + 2$.

(27) Show that the composition of a rotation by $\pi$ about $(0, 0)$ followed by a rotation by $\pi$ about $(2, 0)$ is a translation. Draw pictures.

(28) Find the perpendicular bisector for the segments $\overline{PQ}$ where $P = (1/2, -1/2, 1/2)$, $Q = (2/3, 1/3, -2/3)$.

(29) Draw a frieze pattern with symmetry group consisting just of translations.

(30) Draw a frieze pattern with symmetry group consisting of translations and horizontal reflections and glide reflections only.

(31) What are the symmetry group of the following frieze patterns:

$\cdots$EEEEEEE$\cdots$ $\quad$ $\cdots$TTTTTTT$\cdots$ $\quad$ $\cdots$SSSSSSS$\cdots$