Q11. a) $f(z) = \dfrac{1}{z(z-2)^2}$ centered at $z=2$.

↑ write everything in term of $z-2$.

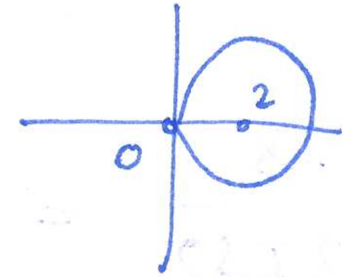
$$= \frac{1}{(z-2)^2} \cdot \frac{1}{z} \qquad \frac{1}{1-z} = 1+z+z^2+\cdots$$

$$= \frac{1}{(z-2)^2} \frac{1}{(z-2)+2} = \frac{1}{(z-2)^2} \frac{1}{2} \frac{1}{1+\frac{z-2}{2}}.$$

$$= \frac{1}{2} \frac{1}{(z-2)^2} \left(1 - \frac{(z-2)}{2} + \left(\frac{z-2}{2}\right)^2 - \left(\frac{z-2}{2}\right)^3 + \cdots \right).$$

$$= \frac{1}{2} \frac{1}{(z-2)^2} - \frac{1}{4} \frac{1}{(z-2)} + \frac{1}{2^3} - \frac{1}{2^4}(z-2)^3 + \cdots$$

converges for $\left|\frac{z-2}{2}\right| < 1$. $|z-2| < 2$.



$$f(z) = \frac{1}{z(z-2)^2} = \frac{1}{z} \cdot \frac{1}{(z-2)^2}.$$

$$\frac{1}{(z-a)\cancel{+a}} \cancel{\frac{1}{}}.$$

$$f(z) = \frac{1}{z(z+1)} \qquad z = a.$$

$$\frac{A}{z} + \frac{B}{z+1}.$$

$$\frac{1}{(z-a)+a} \cdot \frac{1}{(z-a)+(1+a)}. \qquad f(z) = \frac{1}{z} - \frac{1}{z+1}$$

$$f(z) = \frac{1}{(z-a)+a} - \frac{1}{(z-a)+(1+a)}.$$

$$\frac{1}{a} \frac{1}{1+\left(\frac{z-a}{a}\right)} - \frac{1}{1+a} \frac{1}{1+\left(\frac{z-a}{1+a}\right)}.$$

$$\frac{1}{a}\left(1 - \left(\frac{z-a}{a}\right) + \left(\frac{z-a}{a}\right)^2 \cdots\right) - \frac{1}{a+1}\left(1 - \frac{z-a}{1+a} + \left(\frac{z-a}{1+a}\right)^2 + \cdots\right)$$

$$\left|\frac{z-a}{a}\right| < 1 \qquad \left|\frac{z-a}{1+a}\right| < 1$$

$$|z-a| < \min\{|a|, |a+1|\}.$$

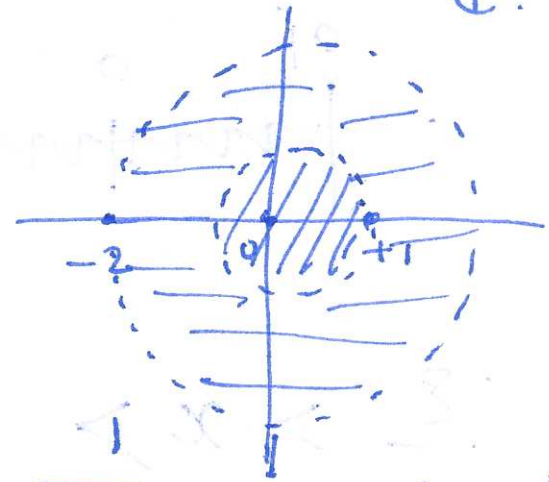
b) $f(z) = \dfrac{z-2}{z+1}$ centred at $z=-1$.

1. $$\frac{z+1-3}{z+1} = 1 - \frac{3}{z+1}$$

2. $$\begin{array}{r|l} & \ \ 1 \\ \hline z+1 & z-2 \\ & z+1 \\ \hline & -3. \end{array}$$

$$1 - \frac{3}{z+1}.$$

c) $f(z) = \dfrac{1}{(1-z)(z+2)}$. $\quad 1 < |z| < 2$.



at zero:

$= \dfrac{A}{1-z} + \dfrac{B}{z+2} = \dfrac{1/3}{1-z} + \dfrac{1/3}{z+2} \Rightarrow \dfrac{1}{6}\dfrac{1}{1+z/2}$.

$$\left.\begin{aligned} &\frac{1}{3}\left(1+z+z^2+z^3+\cdots\right) \\ &+\frac{1}{6}\left(1-\frac{z}{2}+\frac{z^2}{2^2}-\left(\frac{z}{2}\right)^3+\cdots\right) \end{aligned}\right\}$$

radius of convergence

$|z|<2$. $\quad |z|<1$. $\quad |z| < \min\{1,2\}$

$\left|\frac{z}{2}\right|<1$. $\quad 1$.

$\dfrac{1}{3}\dfrac{1}{1-z} = \dfrac{1}{3}\dfrac{1}{z}\dfrac{1}{\frac{1}{z}-1} = -\dfrac{1}{3z}\dfrac{1}{1-\frac{1}{z}}$.

$-\dfrac{1}{3z}\left(1+\dfrac{1}{z}+\dfrac{1}{z^2}+\cdots\right)$ converges $\left|\dfrac{1}{z}\right|<1$ $\quad |z|>1$

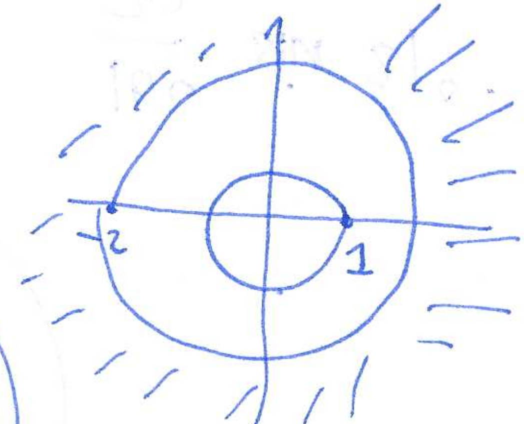
$$-\frac{1}{3z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)+\frac{1}{6}\left(1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+\cdots\right).$$

converges $1<|z|<2$

$|z|>1.$

$$\frac{1}{6}\,\frac{1}{1+z/2}=\frac{1}{6z}\,\frac{1}{\frac{1}{z}+\frac{1}{2}}.$$

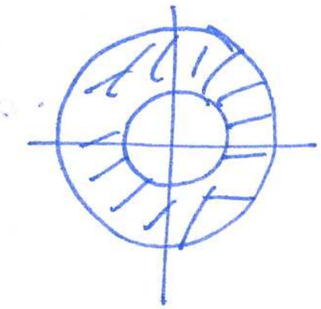
$$=\frac{1}{3z}\,\frac{1}{1+2/z}=\frac{1}{3z}\left(1-\frac{2}{z}+\left(\frac{2}{z}\right)^2+\cdots\right).$$



$\left|\frac{2}{z}\right|<1.$

$|z|>2.$

Q15. $f(z) = z^7 - 4z^3 + 1$



how many zeros in $1 < |z| < 2$?

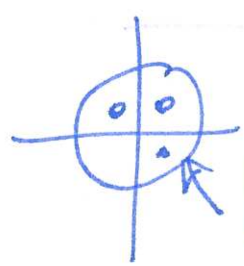
$|z| \le 1$. $\overbrace{z^7 - \underbrace{4z^3}_{f} + 1}^{g}$.

$f(z) = -4z^3$. $|f(z)| = 4$ on $|z| = 1$.

$g(z) = z^7 + 1$ $|g(z)|$ ~~th~~ ≤ 2

$|g(z)| \le |f(z)|$ on $|z| = 1$ so # zeros of f = # zeros of g in $|z| \le 1$.

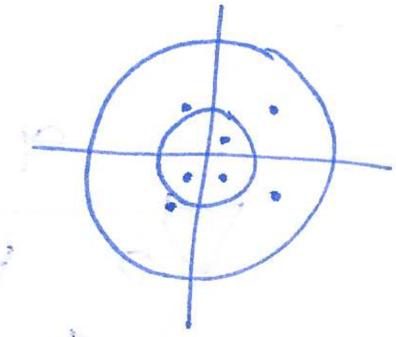
f has 3 zeros.



$|z| = 1$

7

$$f_0(z) = \underbrace{z^7}_{f} - \underbrace{4z^3 + 1}_{g} \qquad |z| = 2.$$



$$|z| = 2 \quad |f(z)| = |z^7| = |z|^7 = 2^7 = 128$$

$$|g(z)| \le |-4z^3| + |1| \le 32 + 1 = 33.$$

$$|g(z)| < |f(z)| \text{ on } |z| = 2.$$

by Rouché : same number of zeros $|z| \le 2$.

z^7 has 7 zeros in $|z| \le 2$.

$\Rightarrow$ there are 4 zeros in $1 \le |z| \le 2$.

Q6 $u(x,y) = ax^2 + bxy + cy^2$. harmonic

$u_{xx} + u_{yy} = 0$

$\frac{\partial u}{\partial x} = 2ax + by$ $\quad \frac{\partial u}{\partial y} = bx + 2ay$

$2a + 2c = 0$

$\frac{\partial^2 u}{\partial x^2} = 2a$ $\quad \frac{\partial^2 u}{\partial y^2} = 2\cancel{a}c$

$a = -c$.

find conjugate $v(x,y)$. $\quad f(z) = u + iv$ analytic.

satisfy the Cauchy-Riemann equations.

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

$\frac{\partial v}{\partial y} = 2ax + by$ $\quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -bx - 2\overset{c}{\cancel{a}}y$.

$v = 2axy + \frac{1}{2}by^2$ $\quad v = -\frac{1}{2}bx^2 - 2\overset{c}{\cancel{a}}xy$

$\textcircled{9}$

$v = 2axy + \frac{1}{2}by^2$ $\quad v = -\frac{1}{2}bx^2 + 2axy$ $\quad a = -c$

$v(x,y) = 2axy + \frac{1}{2}b(y^2 - x^2)$

$u(x,y) = ax^2 + bxy - ay^2$ $\quad \underline{Q}\ f(z) = Az^2\,?$

$$\overset{A}{(\alpha+\beta i)}(x+iy)^2 = (\alpha+\beta i)(x^2 - y^2 + 2xyi)$$

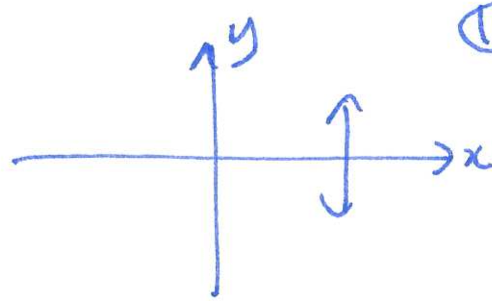
$$= \underbrace{\alpha(x^2-y^2) - 2\beta xy}_{u} + i\underbrace{\left(\alpha 2xy + \beta(x^2-y^2)\right)}_{v}$$

$\boxed{\alpha = a}$ $\quad -2\beta = b$ $\quad \boxed{A = a - \frac{1}{2}bi}$

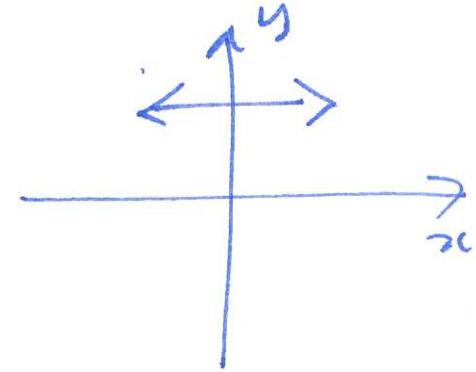
$\boxed{\beta = -\frac{1}{2}b}$ $\quad f(z) = Az^2$

Q4 $f: z \mapsto \bar{z}$

$x+iy \mapsto x-iy$



$\mathbb{C}$.



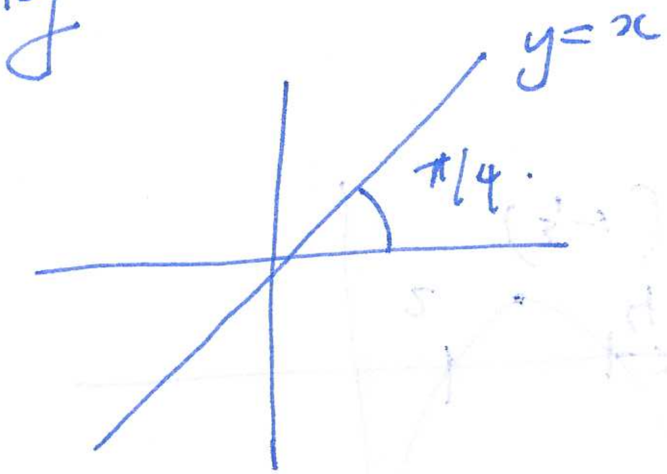
$g: z \mapsto e^{\frac{\pi}{2}i} z = iz.$

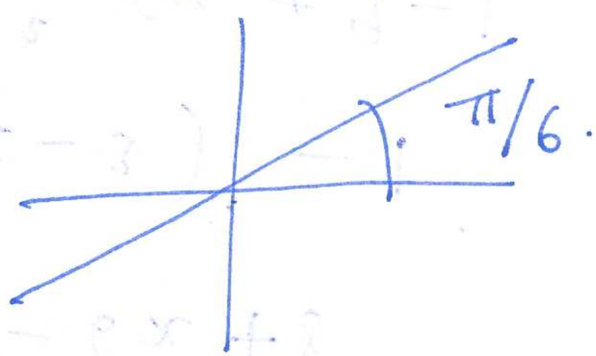
$gfg^{-1}.$

$$z \xmapsto{g^{-1}} -iz \xmapsto{f} \overline{-iz} = i\bar{z} \xmapsto{g} ii\bar{z} = -\bar{z}$$

$$x+iy \mapsto -(x-iy) = -x+iy$$

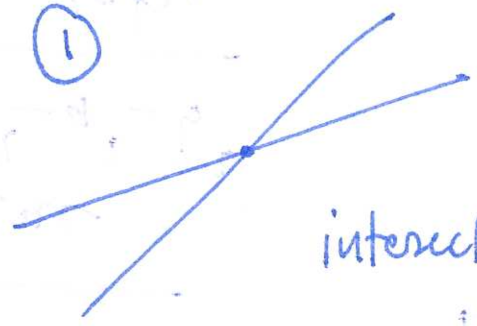
$$z \mapsto -\bar{z}$$





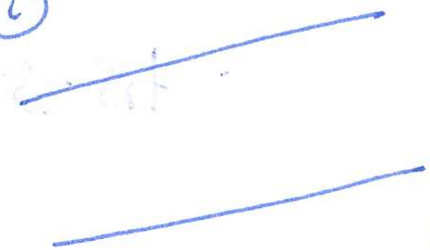
two reflections

1



intersect

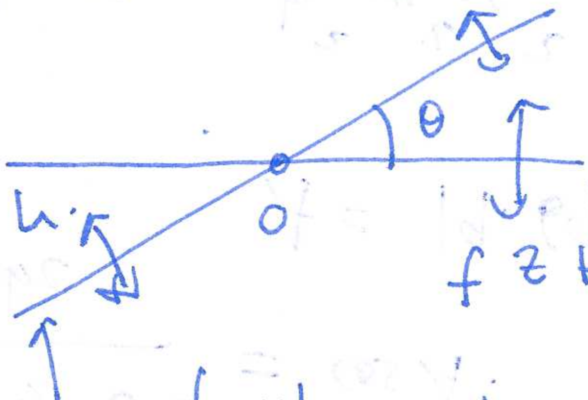
2



parallel.

$\mathbb{C}$.

1



$f\ z \mapsto \bar{z}$

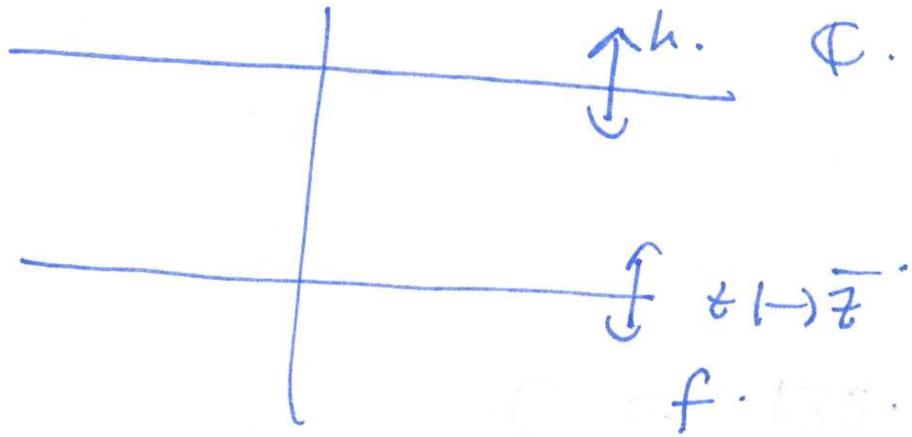
$g f g^{-1}$ where $g: z \mapsto e^{i\theta} z$.

$$z \mapsto e^{-i\theta} z \longmapsto \overline{e^{-i\theta} z} = e^{i\theta}\bar{z} \longmapsto e^{i\theta} e^{i\theta} \bar{z} = e^{2i\theta}\bar{z}$$

$$hf: z \longmapsto \bar{z} \longmapsto e^{2i\theta}\bar{\bar{z}} = e^{2i\theta} z$$

rotation by 2θ

parallel ①.

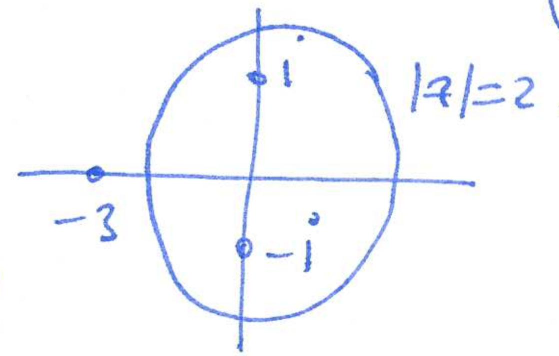


$h = g f g^{-1}$ $\qquad g : z \mapsto z + ai \quad a \in \mathbb{R}_{\geq 0}$.

$h : z \mapsto z - a \mapsto \overline{z - ai} = \bar{z} + ai \longmapsto \bar{z} + ai + ai = \bar{z} + 2ai$.

$hf : z \mapsto \bar{z} \mapsto \bar{\bar{z}} + 2ai = z + 2ai$.

Q14 $\displaystyle\int_C \frac{\sin(z)}{(z+3)(z^2+1)}\,dz$ $\quad C: |z|=2$



Cauchy integral thm: $\displaystyle\int_C f(z)\,dz = 2\pi i \sum_{1} \operatorname{Res} f$

$$\operatorname*{Res}_{i} f = \lim_{z\to i} \frac{\cancel{(z-i)}\sin(z)}{(z+3)\cancel{(z-i)}(z+i)} = \frac{\sin(i)}{(i+3)(2i)}.$$

$$\operatorname*{Res}_{-i} f = \lim_{z\to -i} \frac{\cancel{(z+i)}\sin(z)}{(z+3)(z-i)\cancel{(z+i)}} = \frac{\sin(-i)}{(-i+3)(-2i)}.$$

$$\int_C f(z)\,dz = 2\pi i\left[\frac{\sin(i)}{(i+3)(2i)} + \frac{\sin(-i)}{(-i+3)(-2i)}\right].$$

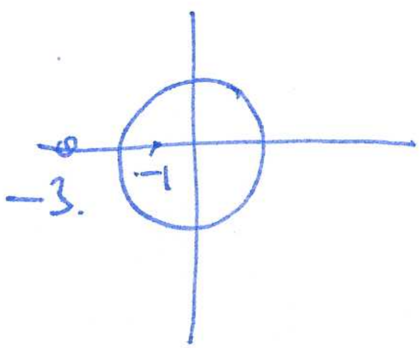
$$\int_C f(z)\,dz = 2\pi i\,\frac{\sin(i)}{2i}\left[\frac{1}{i+3}+\frac{1}{-i+3}\right].$$

$$= \pi\sin(i)\,\frac{\cancel{-i}+3+\cancel{i}+3}{10} = \frac{3\pi}{5}\sin(i).$$

$$= \frac{3\pi}{5}\sinh(1)$$

$$= \frac{3\pi}{5}\left(\frac{e+e^{-1}}{2}\right).$$

<u>Variant</u>: $$\int_C \frac{\sin(z)}{(z+3)(z+1)^2}\,dz.$$

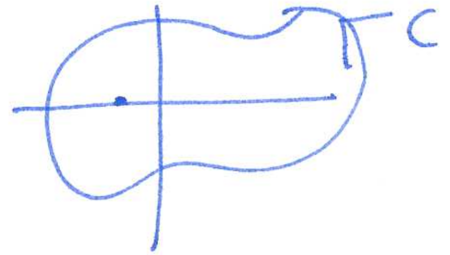


$$\operatorname{Res}_{-1} f = \lim_{z\to -1}\frac{d}{dz}\,\frac{\cancel{(z+1)^2}\sin(z)}{(z+3)\cancel{(z+1)^2}}.$$

$$= \lim_{z\to -1}\frac{d}{dz}\,\frac{\sin(z)}{z+3} = \lim_{z\to -1}\frac{(z+3)\cos(z)-\sin(z)}{(z+3)^2}$$

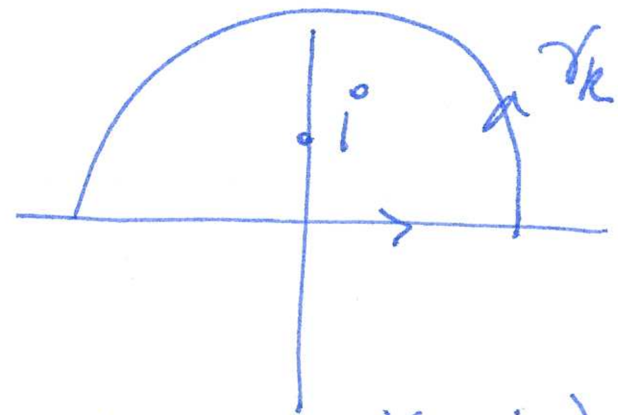
$$= \frac{2\cos(-1)-\sin(-1)}{4}.$$

$$\int_C \frac{\sin(z)}{z+1}\, dz.$$



$$\cancel{\int_0^\infty \frac{\sin(z)}{z}}$$

$$\int_0^\infty \frac{\sin(x)}{x^2+1}\, dx.$$



$$f(z) = e^{iz} = e^{i(x+iy)}.$$

$$= e^{ix-y}$$

$$= e^{-y}\underbrace{(\cos x + i\sin x)}_{|\cdot|=1.}.$$

$|\cdot| \leq 1.$

want $\lim_{R\to\infty} \int_{\gamma_R} f(z)\,dz \longrightarrow 0$

$$\left|\int_{\gamma_R} f(z)\,dz\right| \leq \pi R \cdot \max_{z\in\gamma_R} |f(z)|$$

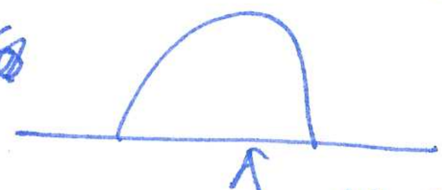
$$\leq \pi R \cdot \frac{1}{R^2-1}. \longrightarrow 0 \text{ as } R\to\infty.$$

$$\frac{\sin(z)}{z^2+1} \qquad \sin(z) = \frac{e^{iz}-e^{-iz}}{2i} \qquad z = x+iy$$

$$= \frac{e^{ix-y} - e^{-ix+y}}{2}$$

$\sin(x)$ $\qquad \cos(x)$.

e^{iz}

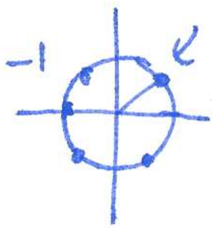


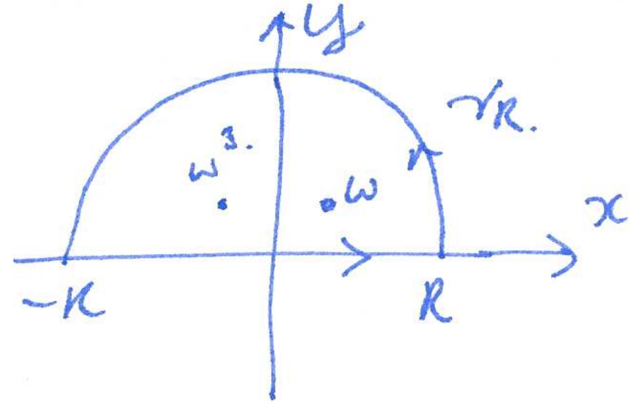
on x-axis.

$$e^{iz} = e^{ix} = \cos x + i\sin x.$$

Q18 $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^4}\,dx.$

$$\frac{1+i}{\sqrt{2}} = \omega.$$





$$f(z) = \frac{e^{iz}}{1+z^4}.$$

$e^{iz} \leq 1$ for $y \geq 0$.

Cauchy int. Thm.

$$\int_C f(z)\,dz = 2\pi i \sum \operatorname{Res} f$$

$$\int_C f(z)\,dz = \underbrace{\int_{\gamma_R} f(z)\,dz}_{\leq \pi R \max_{z \in \gamma_R} |f(z)|} + \underbrace{\int_{-R}^{R} \frac{e^{ix}}{1+x^4}\,dx}_{\int_{-\infty}^{\infty} \frac{\cos x + i \sin x}{1+x^4}\,dx}$$

$$\pi R \frac{1}{R^4 - 1} \to 0 \quad \text{as } R \to \infty.$$

18

$$f(z) = \frac{e^{iz}}{1+z^4} \qquad \operatorname*{Res}_{w} f = \lim_{z\to w} \frac{(z-w)\sin(z)}{1+z^4}.$$

$$\overset{l'H}{=} \lim_{z \to w} \frac{\sin(z) + z\cos(z) - w\cos(z)}{4z^3} = \frac{\sin(w) + \cancel{w\cos(w)} - \cancel{w\cos(w)}}{4w^3}.$$

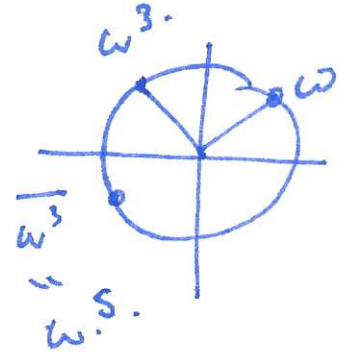
$z\sin(z) - w\sin(z)$

$$\operatorname*{Res}_{w^3} f = \lim_{z\to w^3} \frac{(z-w^3)\sin(z)}{1+z^4}.$$

$$\overset{l'H}{=} \lim_{z\to w^3} \frac{\sin(z) + z\cos(z) - w^3\cos(z)}{4z^3}. = \frac{\sin(w^3) + \cancel{w^3\cos(w^3)} - \cancel{w^3\cos(w^3)}}{4w^9}.$$

$$\int_{-\infty}^{\infty} \frac{\cos x + i\sin x}{1+x^4}\,dx = 2\pi i \left(\frac{\sin(w)}{4w^3} + \frac{\sin(w^3)}{4w^9} \right).$$

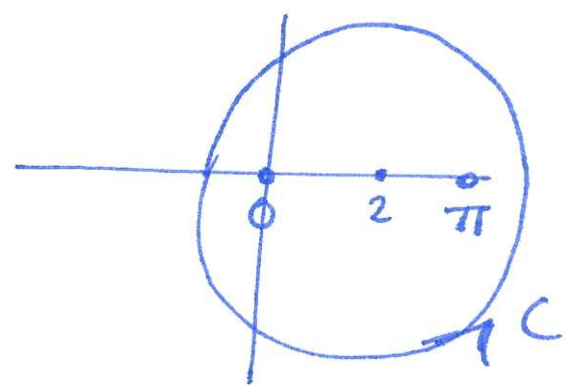
$$\mathrm{Re}\left(2\pi i \left(\frac{\sin(w)\, w^5}{4 \cdot 1} + \frac{\sin(w^3)\, w^7}{4 w^{21} w^7} \right)\right).$$



$$\frac{2\pi i}{2} \left(\sin(w)\, w^5 + \sin(w^3)\, w^7 \right).$$

.... ↑ do in components....

$|z-2| = 3.$



<u>Q16</u> $$\int_C \frac{e^z}{\sin(z)}\, dz = 2\pi i \sum \mathrm{Res}.$$

$\sin(z) = 0 \qquad z = n\pi \quad n \in \mathbb{Z}.$

$$\mathop{\mathrm{Res}}_{0} f = \lim_{z \to 0} \frac{z e^z}{\sin(z)} \overset{l'H}{=} \lim_{z\to 0} \frac{e^z + z e^z}{\cos(z)} = \frac{1}{1} = 1.$$

$$\mathop{\mathrm{Res}}_{\pi} f = \lim_{z \to 0} \frac{(z-\pi) e^z}{\sin(z)} \overset{l'H}{=} \lim_{z\to 0} \frac{e^z + (z-\pi) e^z}{\cos(z)}$$

$$\operatorname{Res}_{\pi} f = \lim_{z\to\pi} \frac{e^z + (z-\pi)e^z}{\cos(z)} = \frac{e^{\pi}}{-1}.$$

$$\int_C \frac{e^z}{\sin z}\,dz = 2\pi i\left(1 - e^{\pi}\right).$$

Q10 a) $f(z) = \dfrac{z}{1+\frac{1}{z}} = \dfrac{z^2}{z+1} = z^2 \dfrac{1}{1+z}.$

$$= z^2\left(1 - z + z^2 - z^3 + \cdots\right) = z^2 - z^3 + z^4 - z^5 + \cdots$$

b) $f(z) = \sin(z^2).$

ux : $\sin(z) = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \cdots$

$$\sin(z^2) = z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \cdots$$

c) $$(\sin(z))^2 = \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)$$

$$\left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 = \frac{e^{2iz} - 2 + e^{-2iz}}{-4} = \frac{1}{2}\left[-\frac{e^{2iz} + e^{-2iz}}{2} + 1\right].$$

$$= \cancel{\frac{1}{2}(}\ \frac{1}{2}(1 - \cos 2z).$$

$$\frac{1}{2}\left(1 - \left(1 - \frac{(2z)^2}{2!} + \frac{(2z)^4}{4!} - \frac{(2z)^6}{6!} + \cdots\right)\right).$$

$$\frac{1}{2}\left(\frac{(2z)^2}{2!} - \frac{(2z)^4}{4!} + \frac{(2z)^6}{6!} - \cdots\right). \quad \sum_{n=1}^{\infty} \frac{1}{2} \frac{2^{2n} z^{2n}}{(2n)!}(-1)^{\#}.$$

$$f(z) = e^{\sin(z)} = e^{\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} \cdots\right)}.$$

$$= 1 + \left(z - \frac{z^3}{3!} + \cdots\right) + \frac{\left(z - \frac{z^3}{3!} + \cdots\right)^2}{2!} + \frac{\left(z - \frac{z^3}{3!} + \cdots\right)^3}{3!} + \cdots$$

$$f'(z) = e^{\sin(z)} \cdot \cos(z).$$

$\vdots$

$$\frac{(z-ia)\cdot e^{iz}}{(z+ia)(z-ia)}.$$

$(z+ia)$