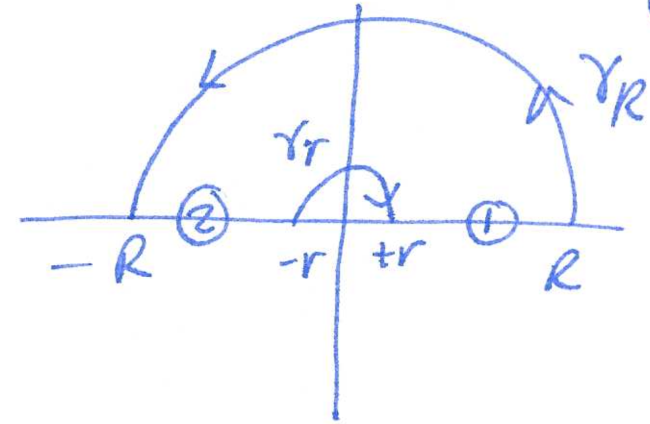
$$\int_0^\infty \frac{\ln(x)^2}{1+x^2}\,dx \qquad f(z) = \frac{\ln(z)^2}{1+z^2}$$

①



① $z(x) = x.$ $\quad \displaystyle\int_r^R \frac{\ln(x)^2}{1+x^2}\,dx = I.$

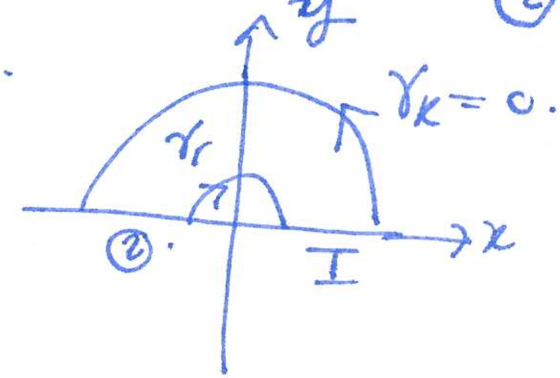
② $z(x) = x$ $\quad \displaystyle\int_{-R}^{-r} \frac{\ln(x)^2}{1+x^2}\,dx = -\int_{-r}^{-R} \frac{\ln(x)^2}{1+x^2}\,dx.$

$x \leftrightarrow -x$

$$+\int_0^\infty \frac{\ln(-x)^2}{1+(-x)^2}\,dx = \int_0^\infty \frac{(\ln(x)+\pi i)^2}{1+x^2}\,dx$$

$$= \int_0^\infty \frac{\ln(x)^2 - \pi^2 + 2\pi i \ln(x)}{1+x^2}\,dx$$

$$= I - \int_0^\infty \frac{\pi^2}{1+x^2}\,dx + 2\pi i \int_0^\infty \frac{\ln(x)}{1+x^2}\,dx.$$



$$\left[\tan^{-1}(x)\right]_0^\infty = \frac{\pi}{2}.$$

$z = \omega.$

for $R >> 0.$

$$= I - \frac{\pi^3}{2} + Im.$$

$$\gamma_R : \left| \int_{\gamma_R} \frac{\ln(z)^2}{1+z^2}\,dz \right| \leq \pi R \frac{(\ln(R)+\theta i)^2}{R^2-1} \longrightarrow 0$$

$\leq \frac{1}{2}\ln(R)^2$

as $R \to \infty$

γ_r : $z(\theta) = re^{i\theta}$

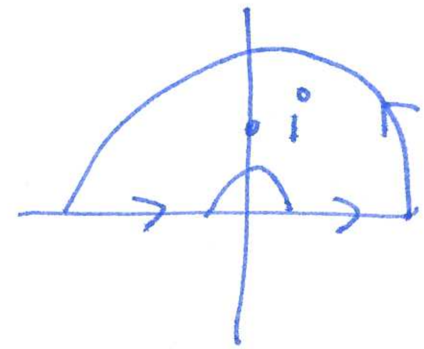
$$\frac{dz}{d\theta} = ire^{i\theta}$$

$$\int_\pi^0 \frac{\ln(re^{i\theta})^2}{1+(re^{i\theta})^2} \cdot ire^{i\theta}\,d\theta.$$

$$\int_{\pi}^{0} \frac{(\ln(r) + o(\theta))^2}{1 + re^{2i\theta}} ire^{i\theta}\, d\theta. \qquad \lim_{r\to 0}.$$

$$\sim \; r\ln(r) \longrightarrow 0 \text{ as } r \longrightarrow 0 \qquad \int_{\gamma_r} \to 0 \text{ as } r \to 0.$$

$$f(z) = \frac{\ln(z)^2}{1+z^2}$$



$$\operatorname*{Res}_{i} f = \lim_{z\to i} \frac{(z-i)\ln(z)^2}{1+z^2}.$$

$$= \lim_{z\to i} \frac{\ln(z)^2}{z+i} = \frac{(\ln(i))^2}{2i} = \frac{(i\pi/2)^2}{2i} = \frac{\pi^2 i}{8}.$$

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{i} f = 2\pi i \left(+ \frac{\pi^2}{8} i \right) = -\frac{\pi^3}{4}.$$

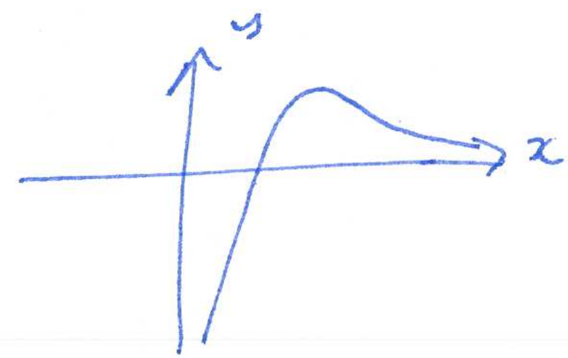
$$\| $$

$$I + I - \frac{\pi^3}{2} + \mathrm{Im} = -\frac{\pi^3}{4}. \quad \leftarrow \text{take real parts.}$$

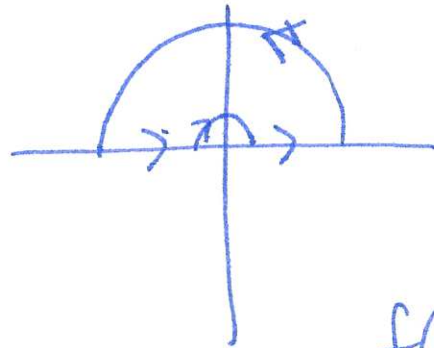
$$2I = -\frac{\pi^3}{4} + \frac{\pi^2}{2} = \frac{\pi^3}{4}$$

$$I = \frac{\pi^3}{8} \qquad \frac{\pi^3}{8}. \leftarrow \text{actual answer.}$$

extra term. $\quad 2\pi i \displaystyle\int_0^\infty \frac{\ln(x)}{1+x^2}\,dx.$



b) $\int_0^\infty \frac{\ln(x)}{(x+a)^2+b^2}\,dx.$

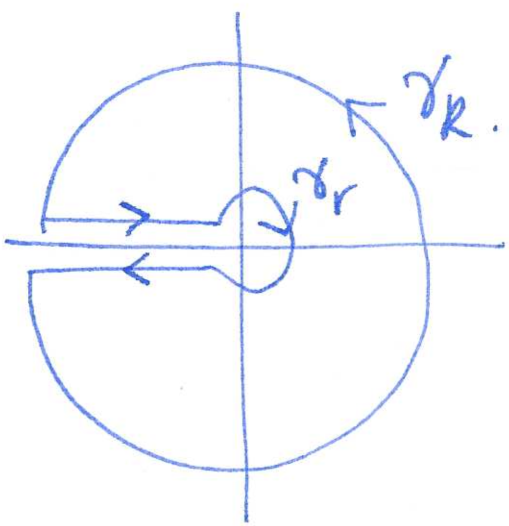


← didn't work for me.

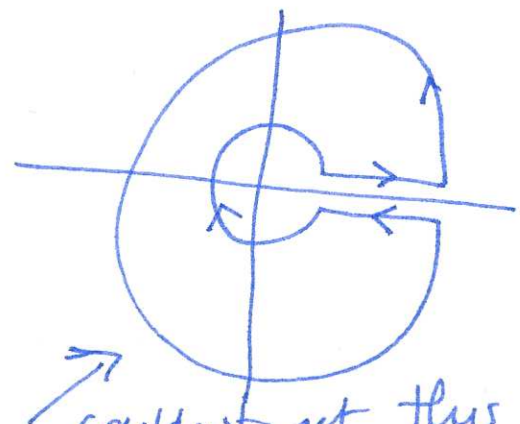
previous: $\frac{\ln(x)^2}{1+x^2}.$

$f(z)= \frac{\ln(z)}{(z+a)^2+b^2}.$

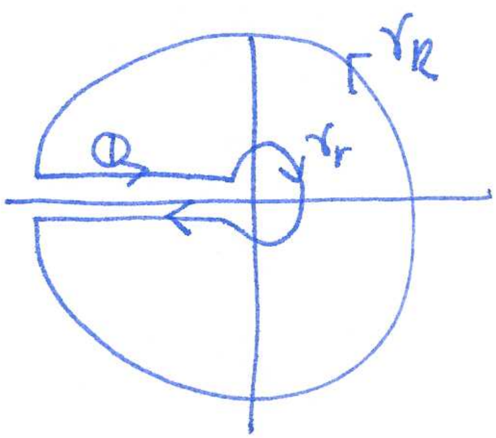
$$\int_0^\infty \frac{\ln(x)+\pi i}{(a-x)^2+b^2}\,dx$$



$f(z)= \frac{\ln(z)^2}{(z-a)^2+b^2}.$



couldn't get this to work.



$$f(z) = \frac{\ln(z)^2}{(z-a)^2+b^2} \quad -\pi < \arg \ln(z) < \pi. \quad ⑥$$

① $z(x) = x + i\epsilon$. $\frac{dz}{dx} = 1$.

$$\int_{-R}^{-r} \frac{\ln(x+i\epsilon)^2}{(x-a)^2+b^2}\,dx = -\int_{-r}^{-R} \frac{\ln(x+i\epsilon)^2}{(x-a)^2+b^2}\,dx.$$

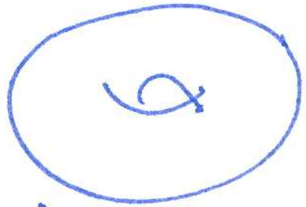
$x \leftrightarrow x$.

$$\int_0^\infty \frac{\ln(-x+i\epsilon)^2}{(-x-a)^2+b^2}\,dx = \int_0^\infty \frac{(\ln(x)+\pi i)^2}{(x+a)^2+b^2}\,dx$$

② $z(x) = x - i\epsilon$.

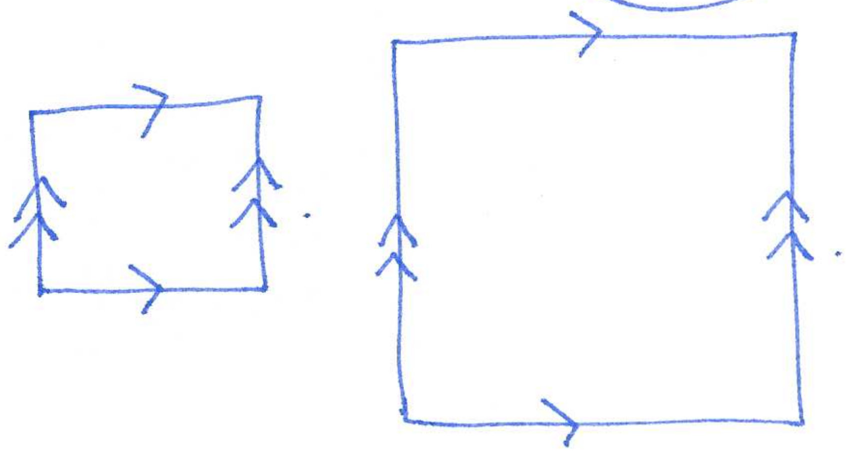
$$-\int_0^\infty \frac{(\ln(x)-\pi i)^2}{(x+a)^2+b^2}\,dx = \int_0^\infty \frac{4\pi i \ln(x)}{(x+a)^2+b^2}\,dx$$

$$= 4\pi i I.$$

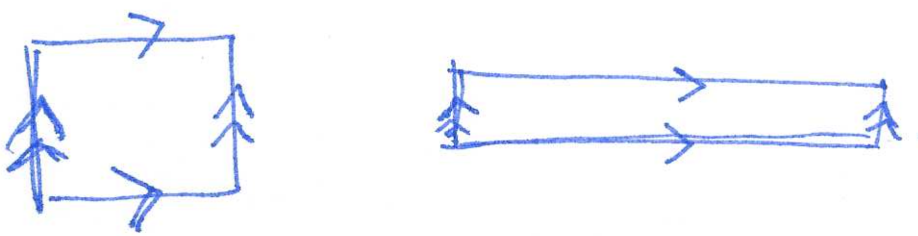
recall. T^2 torus  ← (many)

Euclidean / complex

structures.



Euclidean metrics
up to similarity
$\leftrightarrow$ equivalently rescale to
have area = 1.

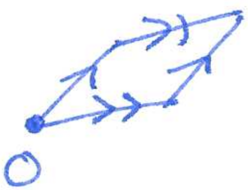


— look at ratio of length of shortest curve to
length of second shortest curve.

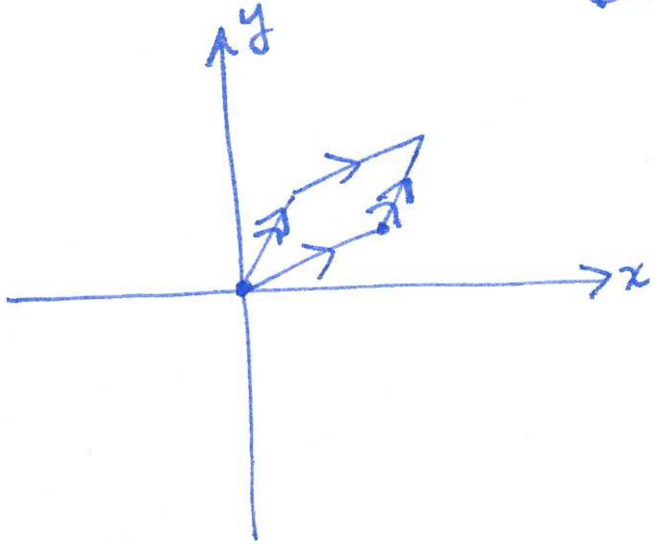
(1)

Q: can we describe all the Euclidean metrics ~~(up to scaling)~~ up to similarity on T^2. A: yes.

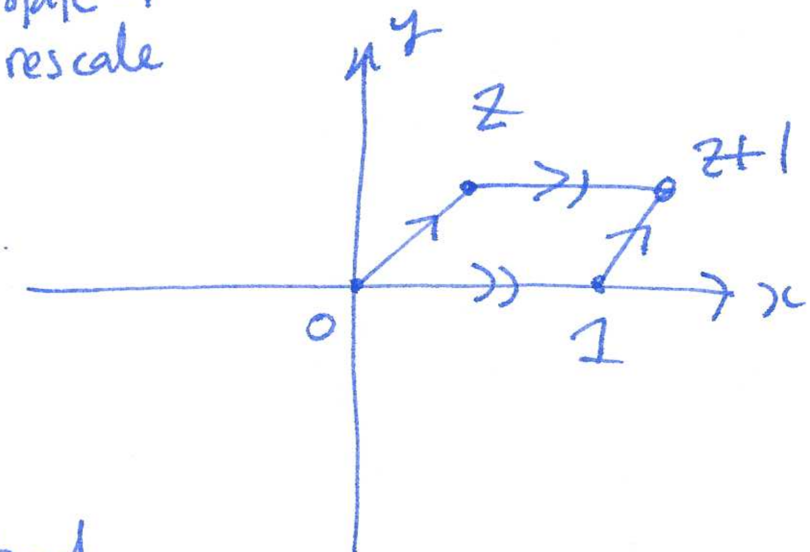
<u>Fact</u>: every Euclidean metric on a torus comes from identifying the sides of a parallelogram.



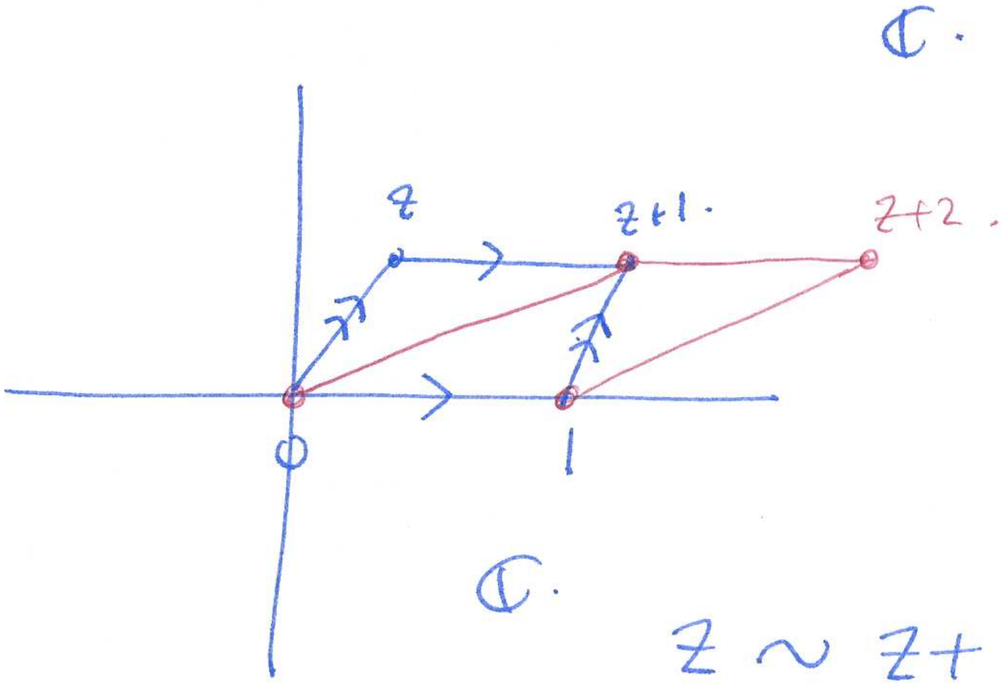
$\mathbb{C}$.

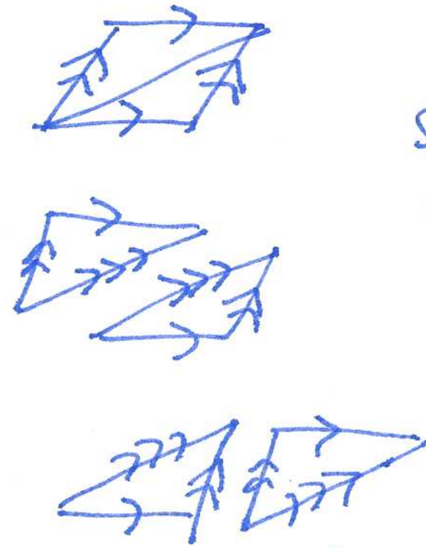


rotate + rescale

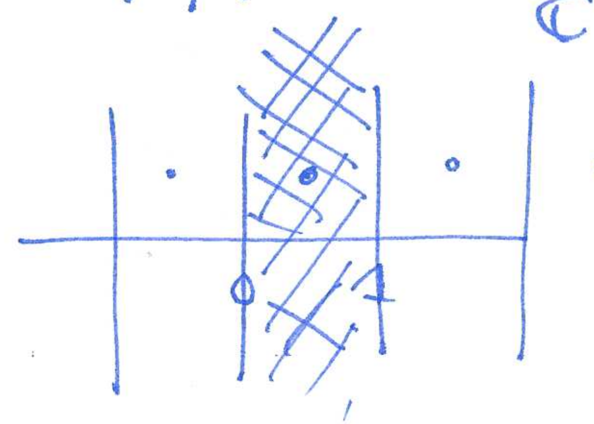


parallelograms $\longleftrightarrow$ parameterized by $z \in \mathbb{C}$.





same Euclidean trl.



$z \sim z+1.$

send z to 1.

$\cancel{z \sim \frac{1}{z}}$.

$\mathbb{C} \longrightarrow \mathbb{C}.$

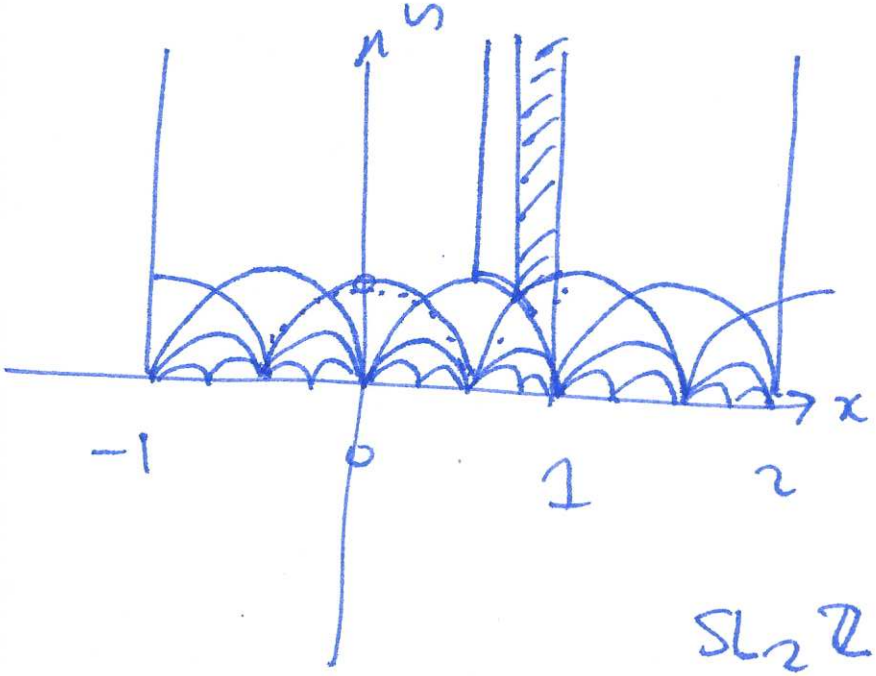
$w \longmapsto \frac{w}{z}.$

$z \mapsto 1.$

$1 \mapsto \frac{1}{z}.$

assume parallelograms live in upper half space

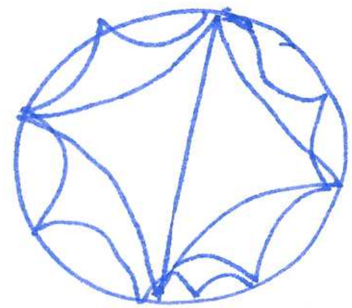
$z \rightsquigarrow -\frac{1}{z}.$



$\mathbb{C}$.

$z \mapsto z+1$.

$z \mapsto -\frac{1}{z}$

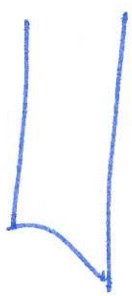


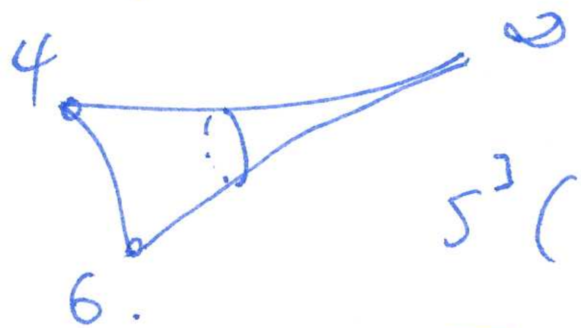
$\mathbb{C}/\sim$.

$SL_2\mathbb{Z} \circlearrowright$ upper half space

via möbius maps.

$\mathbb{C}/\sim$ modular surface. ← explicit parameterization of T^2.





S^2 (3 pts).

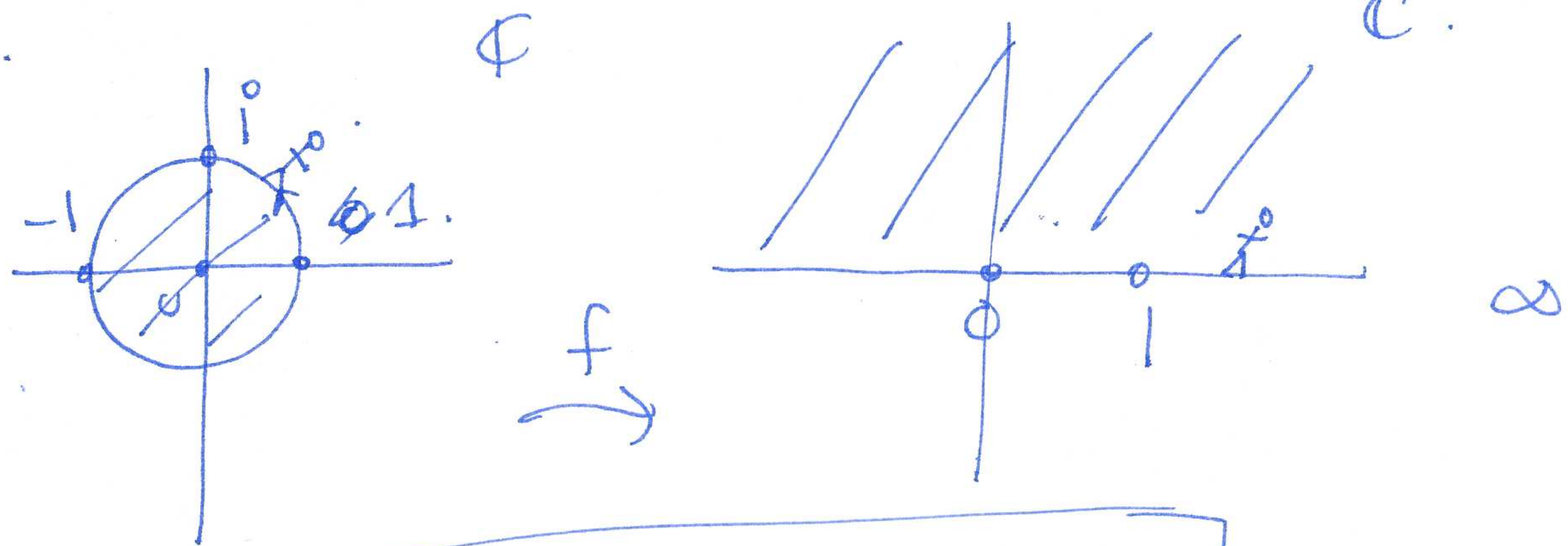
similar story for S_2

← hyp metrics.

dim. $\mathbb{R}^{6g-6}$

S_g ...

Q3.



$1 \mapsto 0$

$i \mapsto 1$

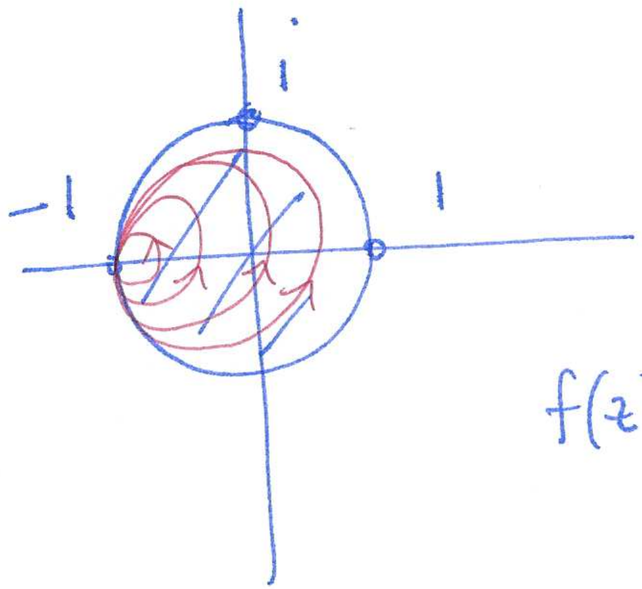
$-1 \mapsto \infty$

$$\boxed{f(z) = \frac{(z-1)}{z+1} \cdot \frac{i+1}{i-1}} = \frac{-i(z-1)}{z+1}$$

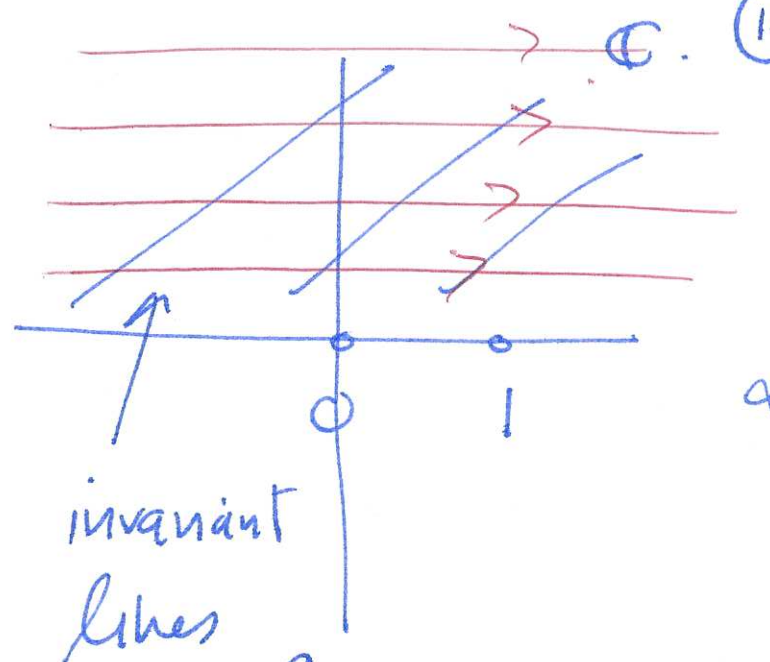
$$f(i) = \frac{i-1}{i+1} \cdot \frac{i+1}{i-1} = 1$$

$$\frac{i+1}{i-1} \cdot \frac{-i-1}{-i-1} = \frac{-1+1-i-i}{2} = -i$$

$$\frac{-iz+i}{z+1}$$



$$\mathbb{C} \underset{\longrightarrow}{\overset{f^{-1}}{\longleftarrow}} \mathbb{C}$$



$$f(z) = \frac{-iz+i}{z+1}$$

$$f^{-1} g f$$

$$z \overset{g}{\longmapsto} z+1$$

$1 \longmapsto 0$

$i \longmapsto 1$

$-1 \longmapsto \infty$

Q6

$$u(x,y) = ax^2 + bxy + cy^2$$

$$\frac{\partial u}{\partial x} = 2ax + by \qquad \frac{\partial u}{\partial y} = bx + 2cy$$

$$\frac{\partial^2 u}{\partial x^2} = 2a \qquad \frac{\partial^2 u}{\partial y^2} = 2c.$$

harmonic $\Leftrightarrow$ $u_{xx} + u_{yy} = 0$.

$\Leftrightarrow$ $2a + 2c = 0$ $\qquad \boxed{a = -c}$.

$u(x,y) = ax^2 + bxy - ay^2.$ Q: find conjugate v. (satisfies Cauchy-Riemann equations)

$\frac{\partial u}{\partial x} = 2ax + by$ $\quad \frac{\partial u}{\partial y} = bx - 2ay$

$v = \int \frac{\partial u}{\partial x} dy$ $\quad v = -\int \frac{\partial u}{\partial y} dx$ $\quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$v = 2axy + \frac{1}{2}by^2 + \alpha(x)$ $\quad v = -\frac{1}{2}bx^2 + 2axy + \alpha(y).$

$v = -\frac{1}{2}bx^2 + 2axy + \frac{1}{2}by^2.$

$\cancel{Az^2}$ $\quad f(z) = u + iv = Az^2.$

$$Az^2 = (\alpha+\beta i)(x+iy)^2.$$

$$= (\alpha+\beta i)(x^2-y^2+2xyi).$$

$$= \underbrace{\alpha(x^2-y^2)-2xy\beta}_{u} + i\underbrace{\left(\beta(x^2-y^2)+2\alpha xy\right)}_{v}.$$

$$a(x^2-y^2) + bxy \qquad -\frac{1}{2}b(x^2-y^2)+2axy$$

choose $a=\alpha$ and $\beta=-\frac{1}{2}b$.

this works for $A=\alpha+\beta i = a-\frac{1}{2}bi$.

Q12 e^z power series centered at $z=-1$.

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \quad \leftarrow \text{centered at } z=0.$$

$$\parallel$$

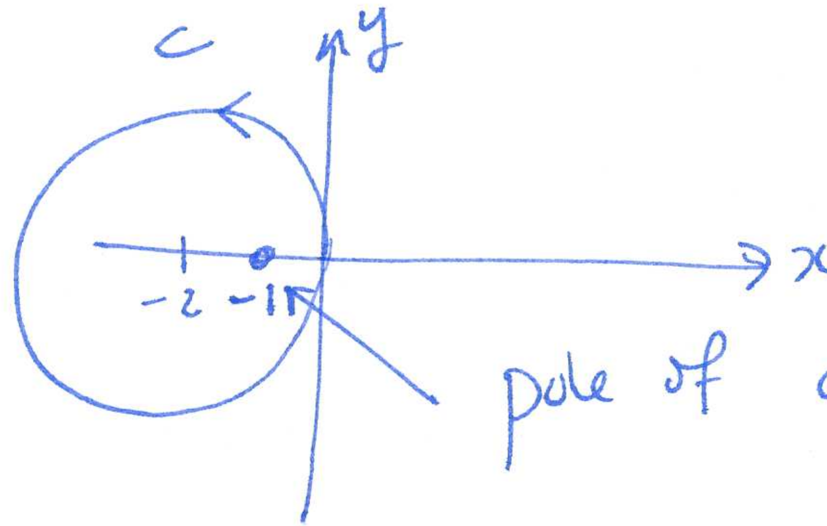
$$e^{z+1-1} = e^{-1} e^{z+1}$$

$$= \frac{1}{e}\left(1 + (z+1) + \frac{(z+1)^2}{2!} + \frac{(z+1)^3}{3!} + \cdots \right).$$

$$\int_C \frac{e^z}{(1+z)^{23}}\,dz$$

$$\parallel$$

$$2\pi i \operatorname*{Res}_{-1} f(z).$$



pole of order 23.

$$f(z) = \frac{e^z}{(1+z)^{23}} \qquad \operatorname*{Res}_{-1} f \leftarrow \text{pole of order 23.}$$

one ~~wrong~~ way:
$$\lim_{z \to -1} \frac{d^{(22)}}{dz^{(22)}} \frac{(1+z)^{23} \cdot e^z}{(1+z)^{23}} \cdot \frac{1}{22!} = \frac{e^{-1}}{22!}.$$

<u>other way</u>

$$\frac{e^z}{(1+z)^{23}} = \frac{1}{(1+z)^{23}} \left(\frac{1}{e} \left(1 + (z+1) + \frac{(z+1)^2}{2!} + \frac{(z+1)^3}{3!} + \cdots \right) \right).$$

$$\operatorname*{Res}_{1} f = \text{coeff on } \frac{1}{z+1}$$

Res.

$$\frac{1}{(1+z)^{23}} \cdot \frac{1}{e} \frac{(z+1)^{22}}{22!} = \frac{1}{e \cdot 22!} \frac{1}{z+1}$$

Q8 $\quad \log(z^2) = 2\log(z)$?

example $\quad z = 1$

$$\log(1^2) = \log(1) = 0 + \boxed{2\pi i n, \ n \in \mathbb{Z}}$$

$$2\log(1) = 2(0 + 2\pi i n) \ n \in \mathbb{Z}$$

$$= 0 + \boxed{4\pi i n, \ n \in \mathbb{Z}}$$

Q7 $|\sin z|^2 = \sin^2 x + \sinh^2 y = \cosh^2 y - \cos^2 x. \quad (z = x+iy)$

$$\| \quad \sin z \cdot \overline{\sin z}$$

$$\cancel{e^z}$$

$$\overline{e^z} = \overline{e^{x+iy}} = \overline{\cancel{\cos}\; e^x \cos y + i e^x \sin y}.$$

$$= e^x \cos y - i e^x \sin y$$

$$= e^{x-iy} = e^{\bar z}$$

$$\overline{\sin z} = \overline{\frac{e^{iz} - e^{-iz}}{2i}} = \frac{\overline{e^{iz}} - \overline{e^{-iz}}}{\overline{2i}}$$

$$= \frac{e^{-i\bar z} - e^{i\bar z}}{-2i} = -\sin(\bar z).$$

$$\sin z \cdot \overline{\sin z} = -\sin z \cdot \sin \bar{z} \qquad z = x+iy \quad \bar{z} = x - iy$$

$$= -\frac{e^{iz}-e^{-iz}}{2i} \cdot \frac{e^{i\bar{z}}-e^{-i\bar{z}}}{2i} \qquad z+\bar{z} = 2x \qquad z-\bar{z} = 2yi.$$

$$= \frac{1}{4}\left[e^{iz+i\bar{z}} - e^{iz-i\bar{z}} - e^{-iz+i\bar{z}} + e^{-iz-i\bar{z}} \right].$$

$$= \frac{1}{4}\left[e^{2ix} + e^{-2ix} - e^{2yi.i} - e^{-2yi.i} \right].$$

$$= \frac{1}{2}\left[\frac{e^{2ix}+e^{-2ix}}{2} - \frac{e^{-2y}+e^{2y}}{2} \right].$$

$$\frac{1}{2}\left(\cos(2x) - \cosh(2y) \right).$$

$$|\sin z|^2 = -\frac{1}{2}\left(\cos(2x) - \cosh(2y)\right)$$

$\cancel{1+}\cosh^2 x - 1$ (21)

$$\cos(2x) = \cos^2 x - \sin^2 x$$
$$= 2\cos^2 x - 1.$$

$$\cosh(2y) = \cosh^2 y + \sinh^2 y$$

$$\frac{e^{2y} + e^{-2y}}{2} \qquad \left(\frac{e^{y}+e^{-y}}{2}\right)^2 + \left(\frac{e^{y}-e^{-y}}{2}\right)^2$$

$$\frac{1}{4}\left(\begin{array}{l} e^{2y} + 1 + e^{-2y} \\ e^{2y} - 1 + e^{-2y} \end{array}\right)$$

$$\frac{1}{2}\left(e^{2y} + e^{-2y}\right)$$

$$\cancel{= \frac{1}{2}\left[2\cos^2 x - 1 - (1 + 2\cosh^2 y)\right].}$$

$$= \frac{1}{2}\left[2\cos^2 x - 1 - \left(2\cosh^2 y - 1\right)\right].$$

$$= -\cos^2 x + \cosh^2 y.$$

$$\cos^2 x + \sin^2 x = 1.$$
$$\cosh^2 x - \sinh^2 x = 1.$$