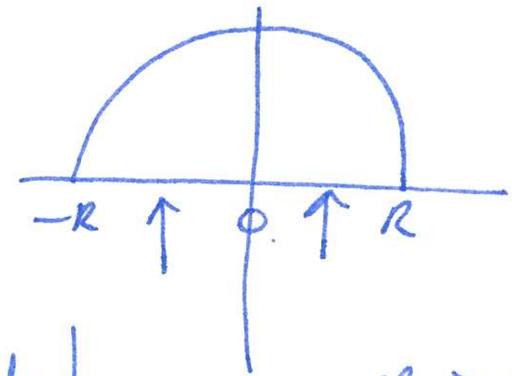
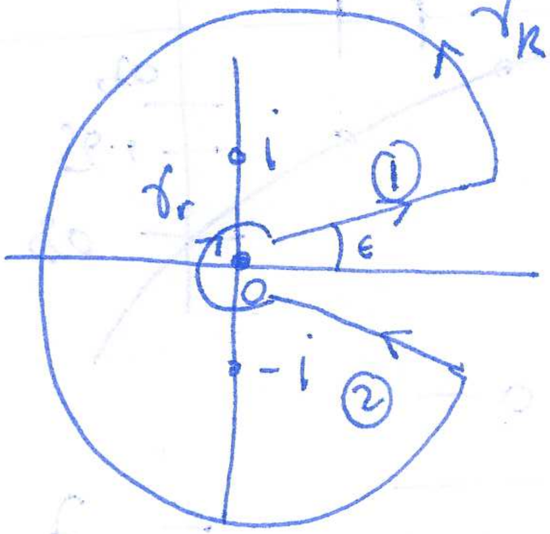
23 d) $\displaystyle\int_0^\infty \frac{x^a}{1+x^2}\,dx$ $\quad -1<a<1 \qquad f(z) = \dfrac{e^{a\ln(z)}}{1+z^2}$





<u>check</u> $\left|\int_{\gamma_R} f(z)\,dz\right| \to 0$ as $R \to \infty$

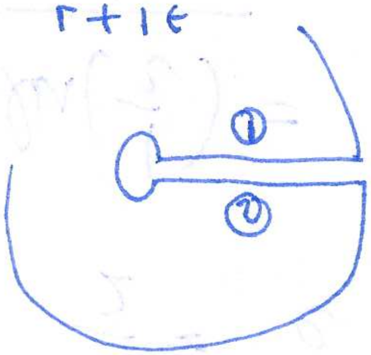
$\left|\int_{\gamma_r} f(z)\,dz\right| \to 0$ as $r \to 0$

① $\cancel{z(r) = re^{i\epsilon} \quad z'(r) = e^{i\epsilon}}$

$z(r) = r + i\epsilon \qquad z'(r) = 1$

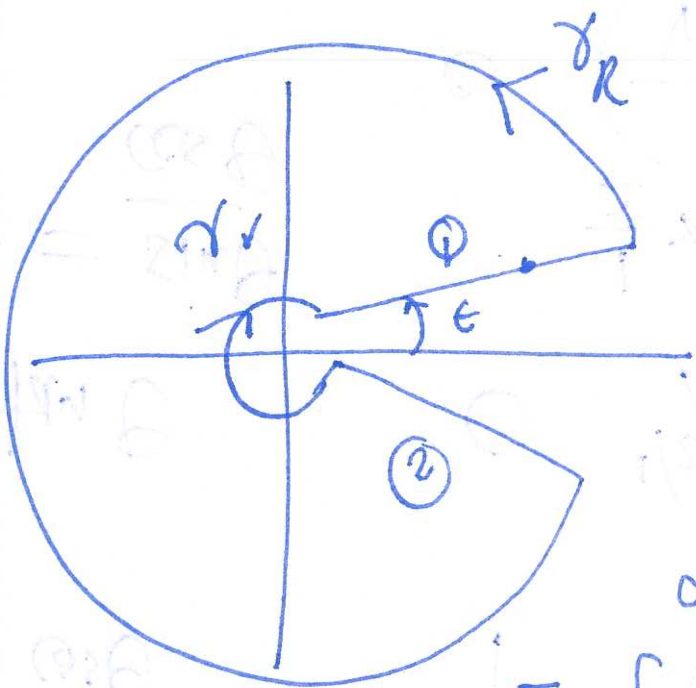
$0 < \arg \ln(z) < 2\pi$

$r + i\epsilon$



$$\cancel{\int_r^R \frac{e^{a\ln(r+i\epsilon)}}{1+(r+i\epsilon)^2}\,dr}$$

$$\cancel{\int_r^R e^a}$$



① $z(r) = re^{i\epsilon}$ $\quad z'(r) = e^{i\epsilon}$. $\quad f(z) = \dfrac{e^{a\ln(z)}}{1+z^2}$. ②

$$\int_r^R \frac{e^{a\ln(re^{i\epsilon})}}{1+(re^{i\epsilon})^2}\, e^{i\epsilon}\, dr.$$

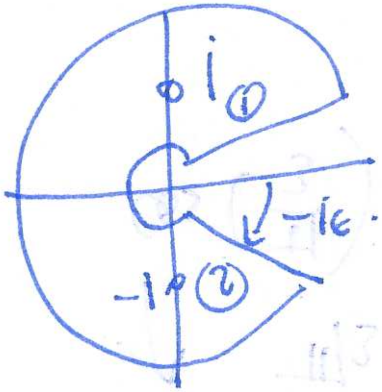
$$= \int_0^\infty \frac{e^{a(\ln r + i\epsilon)}}{1+r^2e^{2i\epsilon}}\, e^{i\epsilon}\, dr.$$

$0 < \ln(z) < 2\pi$

$\epsilon \to 0$

$$= \int_0^\infty \frac{e^{a\ln r}}{1+r^2}\, dr = \int_0^\infty \frac{\cancel{e}\, r^a}{1+r^2}\, dr.$$

$$= I \checkmark.$$



$0 < \arg \ln(z) < 2\pi$

2 $z(r) = \cancel{re^{-i\epsilon}} = re^{i(2\pi-\epsilon)}$.

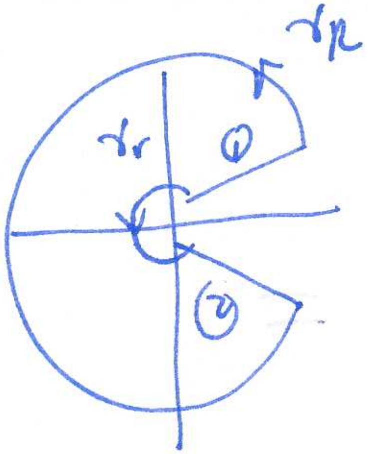
$z'(r) = e^{i(2\pi-\epsilon)}$.

$$\int_r^R \frac{e^{a\ln(re^{-i\epsilon})}}{1+(re^{-i\epsilon})^2} e^{-i\epsilon}\, dr :$$

$$\int_0^\infty \frac{e^{a(\ln(r)+(2\pi-\epsilon)i}}{1+r^2 e^{-i\epsilon.2}} e^{-i\epsilon}\, dr \quad \epsilon \to 0 :$$

$$\int_0^\infty \frac{e^{a(\ln(r)+2\pi i)}}{1+r^2} dr = e^{a2\pi i} \int_0^\infty \frac{r^a}{1+r^2} dr.$$

④



$$\int_C f(z)\,dz = 2\pi i \sum \mathrm{Res}.$$

$$I\left(1-e^{2\pi i a}\right) = 2\pi i \left(\underset{i}{\mathrm{Res}} + \underset{-i}{\mathrm{Res}} \right).$$

Recall **Electrostatics**.

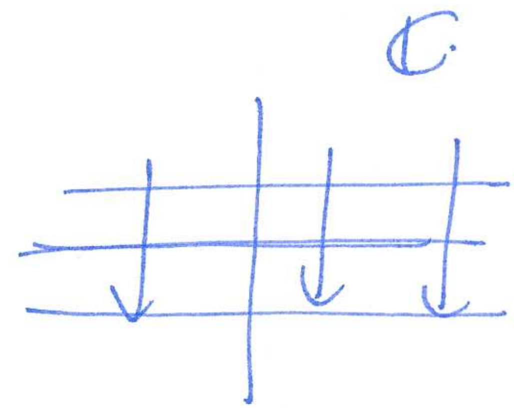
$a \in \mathbb{R}_{>0}$

Example $f(z) = az$ complex potential

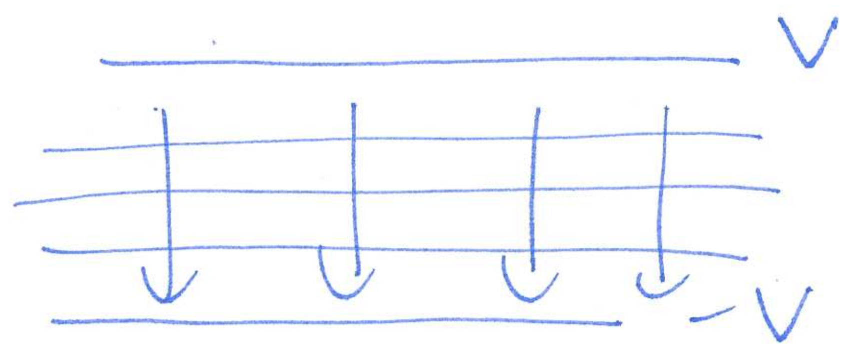
electric field $E = -i\,\overline{f'(z)} = -ia.$

lines of force $ax = \text{const}$

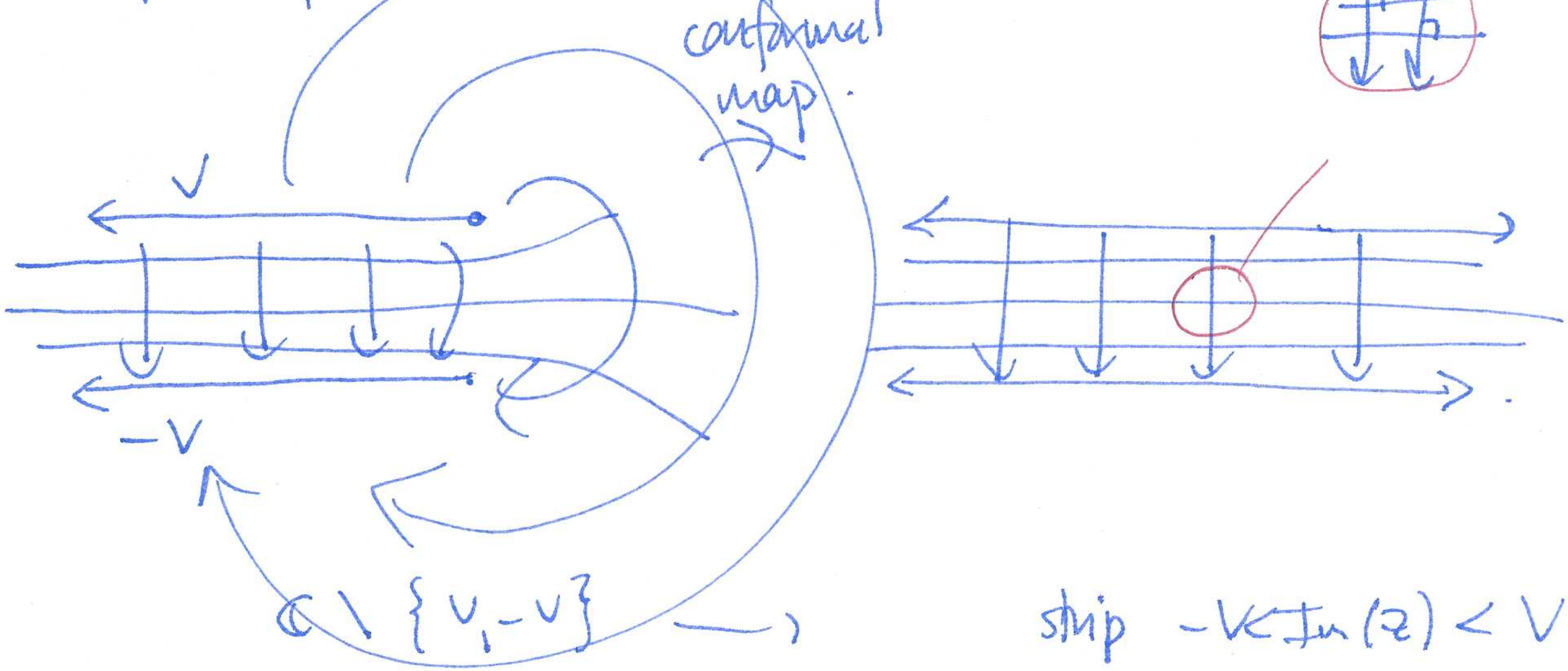
lines of equipotentials $y = \text{const}.$



two parallel conductors



Q: spose parallel conductors not infinite.



conformal map

$$z = \frac{h}{\pi}\left(e^{\pi w/V} + \frac{\pi w}{V}\right).$$

Fibonacci numbers $\quad a_0 = 0 \quad a_1 = 1 \qquad a_n = a_{n-1} + a_{n-2}$

$n \geq 2$.

$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$

set $\quad f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad$ Q: does this converge?

consider $\sum_{n=0}^{\infty} |a_n z^n| \leftarrow$ positive, apply ratio test

$$\lim_{n\to\infty} \frac{|a_{n+1} z^{n+1}|}{|a_n z^n|} = \lim_{n\to\infty} |z| \left| \frac{a_{n+1}}{a_n} \right|.$$

< 1.

$$= \lim_{n\to\infty} |z| \left| \frac{a_n + a_{n-1}}{a_n} \right| = \lim_{n\to\infty} |z| \left| 1 + \frac{a_{n-1}}{a_n} \right| \leq 2|z|$$

$f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges for $2|z| \leq 1$.

$|z| \leq \frac{1}{2}$.

radius of convergence is $r \geq \frac{1}{2}$.

$a_0 = 0$, $a_1 = 1$; $a_n = a_{n-1} + a_{n-2}$.

Q what is $f(z)$?

$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots$$

$$z f(z) = a_0 z + a_1 z^2 + a_2 z^3 + a_3 z^4 + \cdots$$

$$(f + zf) = (a_0 + a_1) z + (a_1 + a_2) z^2 + (a_2 + a_3) z^3 + \cdots$$

$$f + zf = a_2 z + a_3 z^2 + a_4 z^{\cancel{4}\,3} + \cdots$$

$$= \frac{f}{z} - 1.$$

$$f + zf = \frac{f}{z} - 1$$

$$f\left(1 + z - \frac{1}{z}\right) = -1$$

$$f(z + z^2 - 1) = -z.$$

$$\boxed{f(z) = \frac{z}{1 - z - z^2}.}$$

Q: how do we recover the a_n from $f(z)$?

$$\operatorname*{Res}_{z=0} \frac{f(z)}{z^n}$$

$$= \operatorname*{Res}_{0} \frac{z}{z^n(1-z-z^2)} = \frac{1}{z^n}\left(\frac{z}{1-z-z^2}\right).$$

$$\cancel{= \operatorname*{Res}_{0} \frac{1}{z^{n-1}(1-z-z^2)}}$$

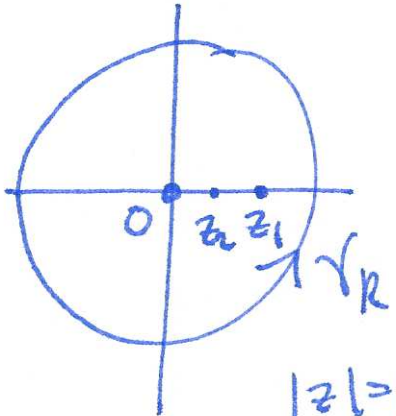
$$= \operatorname*{Res}_{0} \frac{1}{z^n}\left(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots\right)$$

$\frac{1}{z}$ term.

$$\frac{a_{n-1} z^{n-1}}{z^n}$$

$$= a_{n-1}.$$

1.



$|z|=R.$

$$\left|\int_{\gamma_R}\frac{z}{z^n(1-z-z^2)}\,dz\right| \le 2\pi R\,\frac{R}{R^n(R^2-R-1)}.$$

$\to 0$ ~~for~~ as $R \to \infty$.

$$\int_{\gamma_R}\frac{f(z)}{z^n}\,dz = 0 = 2\pi i \sum \mathrm{Res} = 0.$$

$$-(z^2+z-1)=0 \qquad z=\frac{-1\pm\sqrt{1+4}}{2}=\frac{-1\pm\sqrt{5}}{2}. \quad \overset{+}{z_1},\ \overset{-}{z_2}$$

$$\underbrace{\operatorname*{Res}_{0}\frac{f}{z^n}}_{a_{n-1}} + \operatorname*{Res}_{z_1}\frac{f}{z^n} + \operatorname*{Res}_{z_2}\frac{f}{z^n} = 0$$

$$\frac{f(z)}{z^n} = \frac{z}{z^n(1-z-z^2)} \qquad \operatorname*{Res}_{z_1} f \qquad z_1 = \frac{-1+\sqrt{5}}{2}. \quad z_2 = \frac{-1-\sqrt{5}}{2}$$

$$\lim_{z\to z_1} -\frac{(z-z_1)\, z}{z^n(\cancel{z}-z_1)(z-z_2)} = \lim_{z\to z_1} \frac{-1}{z^{n-1}(z-z_2)} \qquad \frac{-1}{z_1^{n-1}(z_1-z_2)}$$

$$\operatorname*{Res}_{z_2} f = \frac{-1}{z_2^{n-1}(z_2-z_1)}. \qquad \sum \operatorname{Res} = 0.$$

$$\cancel{z_2}\ z_1 - z_2 = \cancel{\sqrt{5}}.\ \sqrt{5}.$$

$$a_{n-1} = \frac{1}{z_1^{n-1}(z_1-z_2)} + \frac{1}{z_2^{n-1}(z_2-z_1)}$$

$$a_{n-1}. = \frac{1}{\left(\frac{-1+\sqrt{5}}{2}\right)^{n-1}\sqrt{5}} - \frac{1}{\left(\frac{-1-\sqrt{5}}{2}\right)^{n-1}\sqrt{5}}.$$

$a_0 = 0$

$a_1 = 1.$

$a_2 = 1.$

$a_n = a_{n-1} + a_{n-2} + a_{n-3}.$

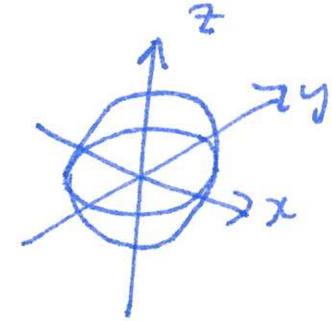
Geometry

Euclidean geometry

Euclids axioms:
1) can construct a line between any two points
2) can extend any line segment indefinitely
3) can construct a circle w/ given center and radius.
4) all right angle are equal.
5) non-parallel lines intersect. ← this is independent of the other 4.

spherical geometry

$S^2 \subseteq \mathbb{R}^3$ $\quad x^2+y^2+z^2=1$



try: points ⟷ points.

lines ⟷ ~~arcs~~ great circles

better: points ⟷ antipodal points
lines ⟷ great arcs.
} projective geometry

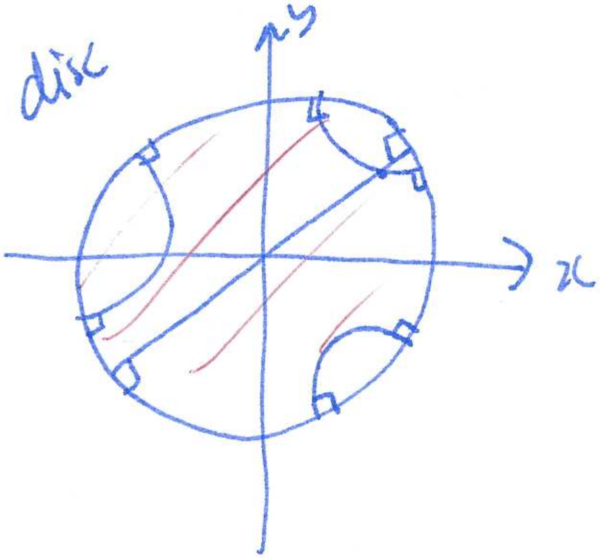
<u>hyperbolic geometry</u> ← satisfies axioms 1–4, but not 5.

two models. disc model: points.

lines – straight lines/circles perpendicular to the boundary

$\mathbb{C}$.

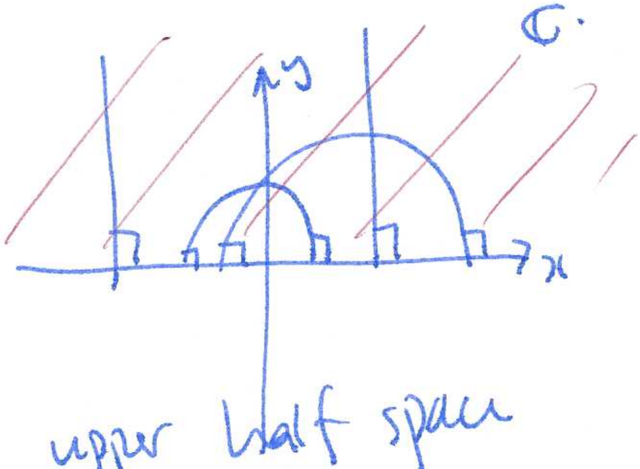
disc



upper half space model.

lines – straight lines/circles perpendicular to the boundary

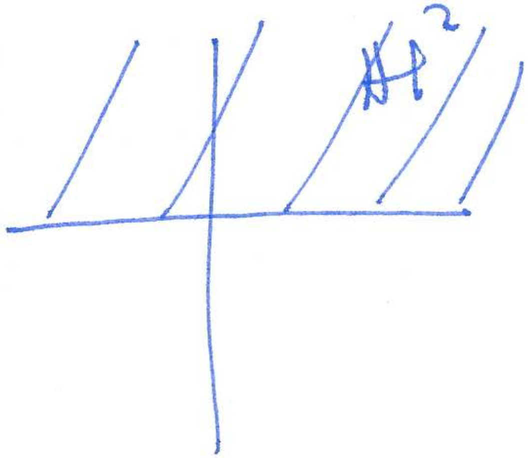
$\mathbb{C}$.



upper half space

↑ these two models give the same geometry: $\exists$ Möbius map which takes the disc to the upper half space – conformal, ~~takes~~ preserves $\left\{\begin{matrix}\text{circles} \\ \text{lines}\end{matrix}\right\}$

key fact: there is a natural metric on this space and the Möbius maps that fix disc/upper half space act as isometries on this space.



← Q: which Möbius maps preserve upper half space?

$$\frac{az+b}{cz+d} \qquad a, b, c, d \in \mathbb{R}.$$

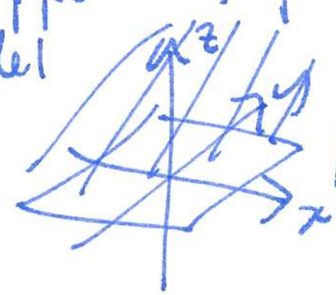
$$\leftrightarrow \quad SL_2\mathbb{R}.$$

→ 3d version of $\mathbb{H}^2$.

ball model → upper half space model

$\mathbb{H}^3$.

∂ ball $= S^2$



$\mathbb{R}^3 = \mathbb{C} \cup \{\infty\}$.

$z > 0$.

<u>Fact</u> $\mathrm{Isom}^+(\mathbb{H}^3)$.

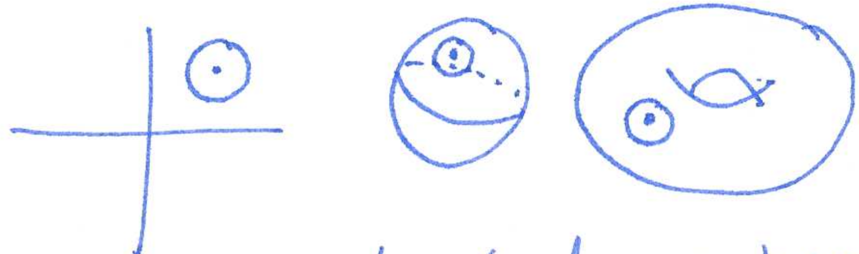
$\|$

Möbius map

$SL_2\mathbb{C}$.

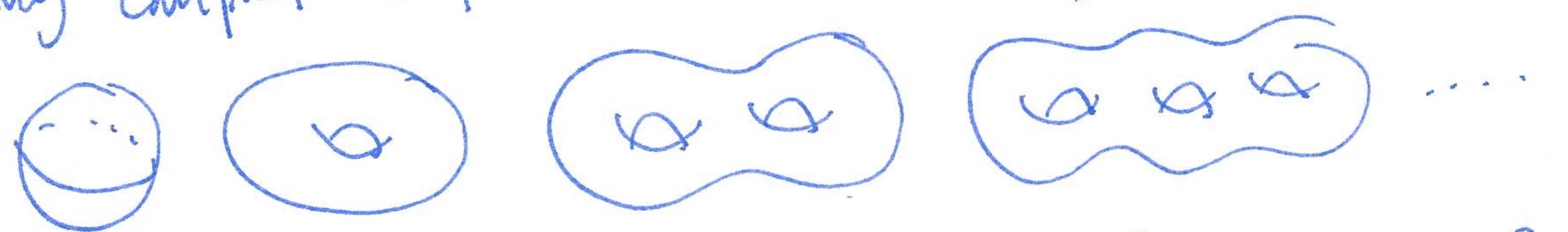
Defn A **surface** is a space which is locally homeomorphic to an open disc.

Examples $\mathbb{R}^2$, $\mathbb{C}$



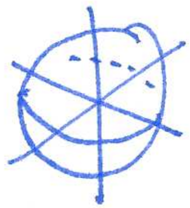
Thm (Classification of compact, closed = no boundary surfaces)

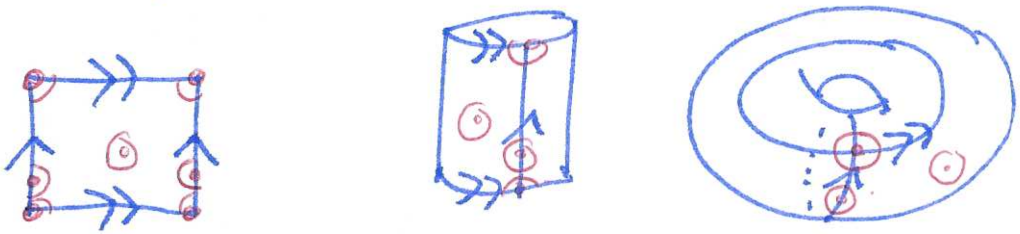
Every compact surface with no boundary is one of:



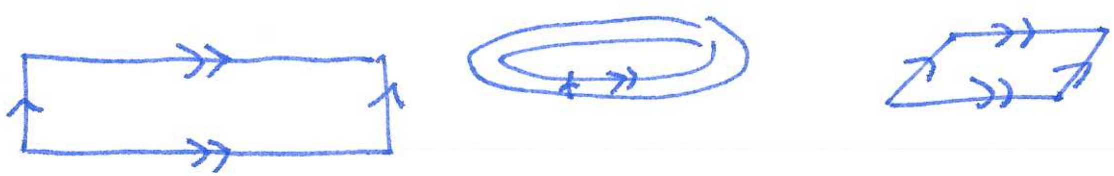
S^2	T^2	S_2	S_3	S_g
$g=0$	$g=1$	genus $g=2$	$g=3$	genus g

Fact: each surface has a geometry associated with it.

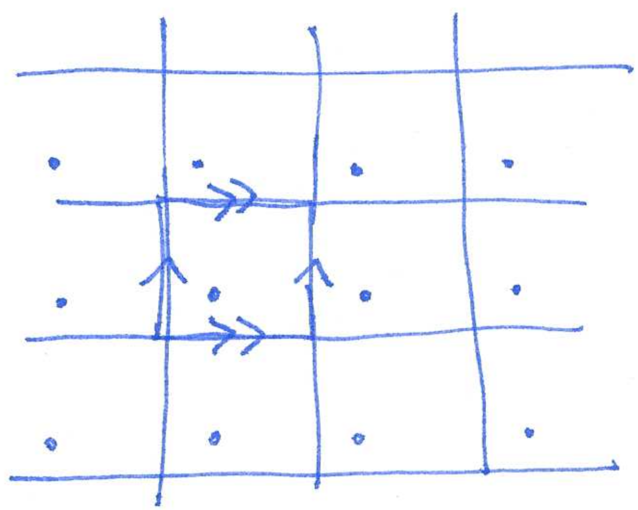
S^2:  ✓ ← only 1 round metric on S^2.

T^2:  ← Euclidean metric on T^2

$[0,1] \times [0,1] \subseteq \mathbb{R}^2$.

Example  parallelogram.

Alternative construction



$\mathbb{R}^2 \leftrightarrow \mathbb{C}$ / $\sim$ ⇝ torus.

$z \mapsto z+1$

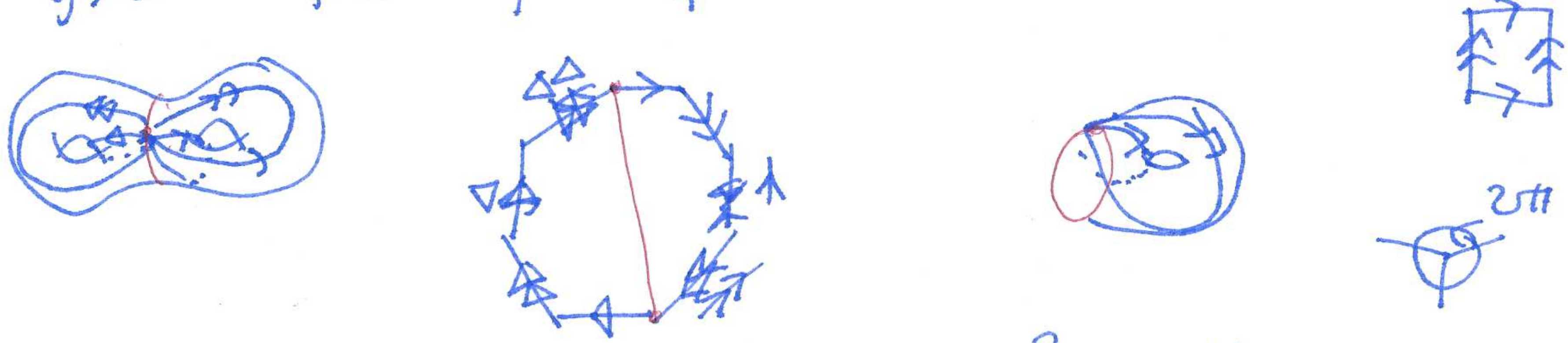
$z \mapsto z+i$

metrics on T^2 $\leftrightarrow$ tilings of $\mathbb{R}^2$

$\mathbb{C}$.

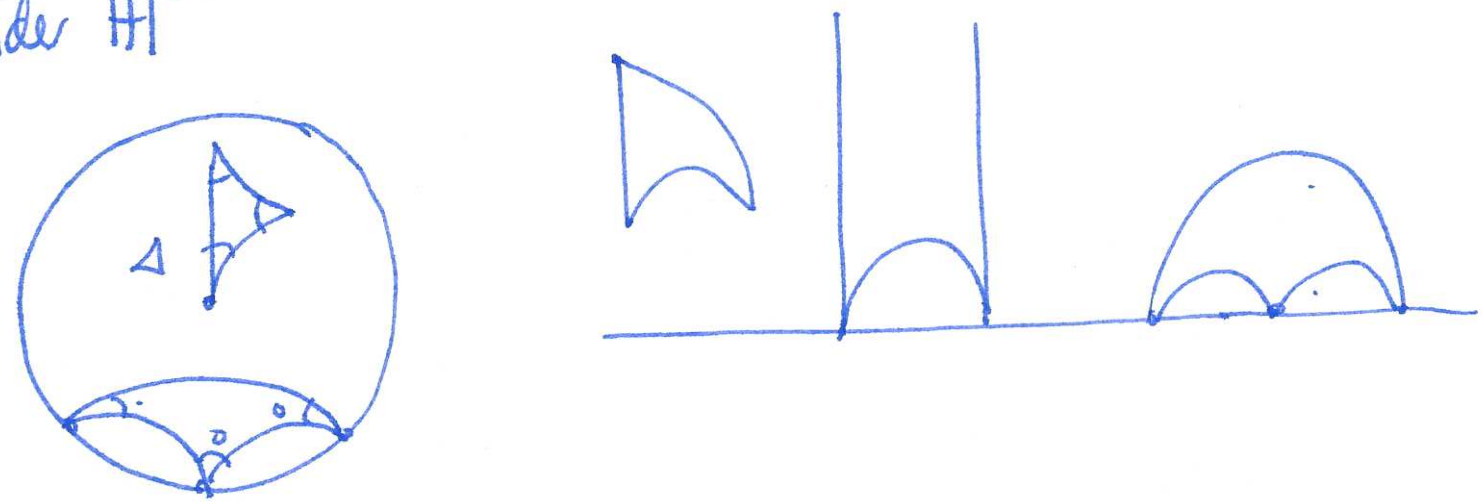
S_g $g \geq 2$: fact: get hyperbolic metrics on these.

S_2

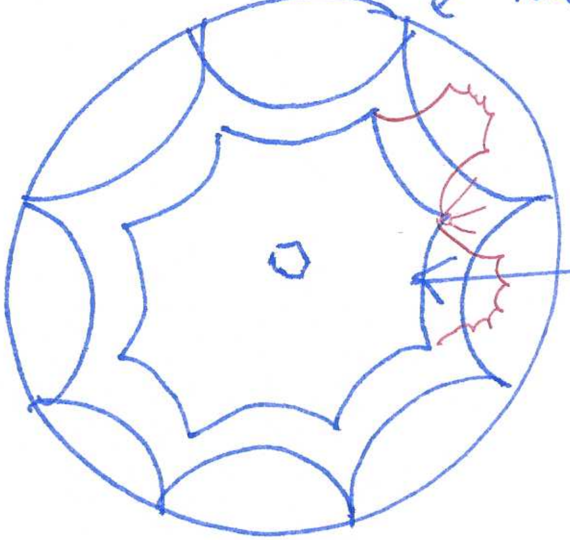


Q: can we tile the plane with octagons? A: No.

consider $\mathbb{H}^2$.



← ideal regular octagon angles $= 0$.



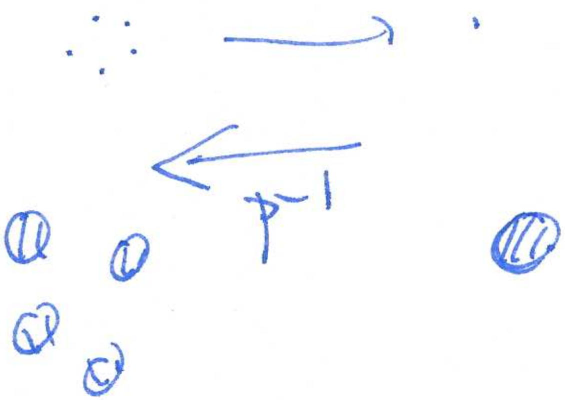
octagon w/ angles $\frac{2\pi}{8} = \frac{\pi}{4}$.

these tile the $\mathbb{H}^2$ plane and give a hyp metric to S_2

<u>Recall</u> $f(z) = p(z)$ ← polynomial of deg n

$\mathbb{C} \cup \{\infty\} \xrightarrow{p} \mathbb{C} \cup \{\infty\}$ inverse multivalued

n-to one (n-valued)



compact no boundary branched cover of S^2 ⟷ S^2 compact no boundary

↑ surface.