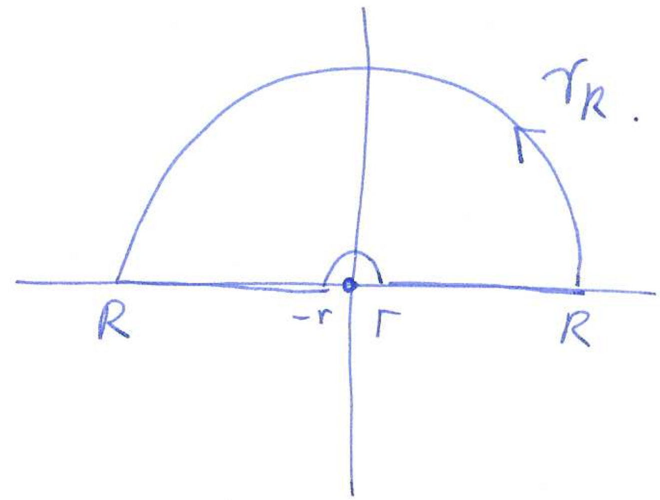
§12 Q16 $\quad \displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx.$



<u>hint</u> $\quad f(z) = \dfrac{e^{2iz} - 1}{z^2}.$

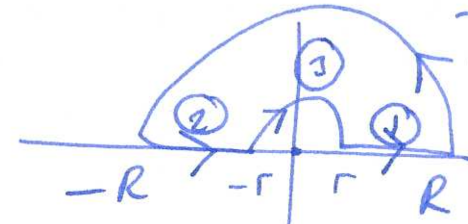
$$\int_C f(z)\,dz = 0$$

check : $\quad \left| \int_{\gamma_R} f(z)\,dz \right| \longrightarrow 0$ as $R \to \infty$.

$$e^{2i(x+iy)} = e^{2(-y+ix)} = \underbrace{e^{-2y}}_{\leq 1.}\ \underbrace{e^{2ix}}_{1.}$$

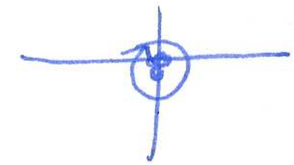
$$\left| \int_{\gamma_R} f(z)\,dz \right| \leq \pi R \cdot \frac{2}{R^2} = \frac{2\pi}{R} \longrightarrow 0 \text{ as } R \to \infty.$$

$$f(z) = \frac{e^{2iz} - 1}{z^2}$$



$$\int_r^R \frac{e^{2iz} - 1}{z^2}\, dz = \int_0^\infty \frac{e^{2ix} - 1}{x^2}\, dx \quad (1)$$

$z = x + i0$.



$$\int_{-R}^{-r} \frac{e^{2ix} - 1}{x^2}\, dx = -\int_{-r}^{-R} \frac{e^{2ix} - 1}{x^2}\, dx$$

$x \longleftrightarrow -x$.

$-x \cancel{\longleftrightarrow} = w$.

$\frac{dw}{dx} = -1$.

$$= \int_r^R \frac{e^{-2ix} - 1}{x^2}\, dx \quad (2)$$

(1) + (2)

$$\int_0^\infty \frac{e^{2ix} - 2 + e^{-2ix}}{x^2}\, dx = \int_0^\infty \frac{\left(e^{ix} - e^{-ix}\right)^2}{x^2}\, dx.$$

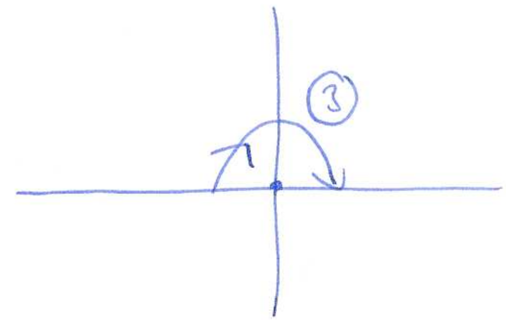
$$\int_0^\infty \frac{(e^{ix}-e^{-ix})^2}{x^2}\,dx \qquad \frac{e^{ix}-e^{-ix}}{2i} = \sin x$$

$$\int_0^\infty \frac{(2i\sin x)^2}{x^2}\,dx = \int_0^\infty -4\frac{\sin^2 x}{x^2}\,dx = -4\,I.$$

$$f(z) = \frac{e^{2iz}-1}{z^2} \qquad z(\theta) = re^{i\theta}$$

$$\theta \in [\pi, 0].$$

$$z'(\theta) = ire^{i\theta}.$$



$$\int_\pi^0 \frac{e^{2ire^{i\theta}}-1}{(re^{i\theta})^2}\, ire^{i\theta}\,d\theta = -\int_0^\pi \frac{e^{2ire^{i\theta}}-1}{re^{i\theta}}\, i\,d\theta$$

$$e^z = 1+z+\frac{z^2}{2!}+\cdots$$

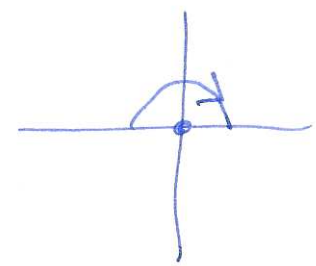
$$-\int_0^\pi \frac{e^{2ire^{i\theta}}-1}{re^{i\theta}}\, i\,d\theta = -\int_0^\pi \frac{\cancel{1} + 2ire^{i\theta} + \frac{(2ire^{i\theta})^2}{2!} + \dots \cancel{-1}}{re^{i\theta}}\, i\,d\theta$$

$$= +\int_0^\pi \underbrace{2i.i}_{-1} + \underbrace{\frac{4}{2!}(i)^2 re^{i\theta}}_{\to 0} + \underbrace{O(r^2)}_{\to 0}\, d\theta = \int_0^\pi 2\,d\theta$$

$$= 2\pi.$$

<u>Aside</u> $\operatorname{Res}_0 f(z) = \lim_{z\to 0} z\,\frac{e^{2iz}-1}{z^2}$.

$$= \lim_{z\to 0} \frac{e^{2iz}-1}{z} \overset{l'H}{=} \lim_{z\to 0} \frac{2ie^{2iz}}{1} = 2i.$$



$$2i \times \frac{2\pi i}{2} = -2\pi.$$

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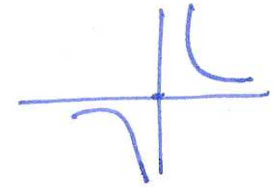
③ + ① $\cancel{4}$ + ② $= 0$

$$2\pi - 4I = 0.$$

$$I = \frac{2\pi}{4} = \frac{\pi}{2} = \int_0^\infty \frac{\sin^2 x}{x^2}\,dx.$$

Q23 e). Cauchy Principal Value.



$$\int_{-1}^{1} \frac{1}{x}\,dx = \underbrace{\lim_{t \to 0-} \int_{-1}^{t} \frac{1}{x}\,dx}_{[\ln|x|]_{-1}^{t}} + \underbrace{\lim_{s \to 0+} \int_{s}^{1} \frac{1}{x}\,dx}$$

DNE.

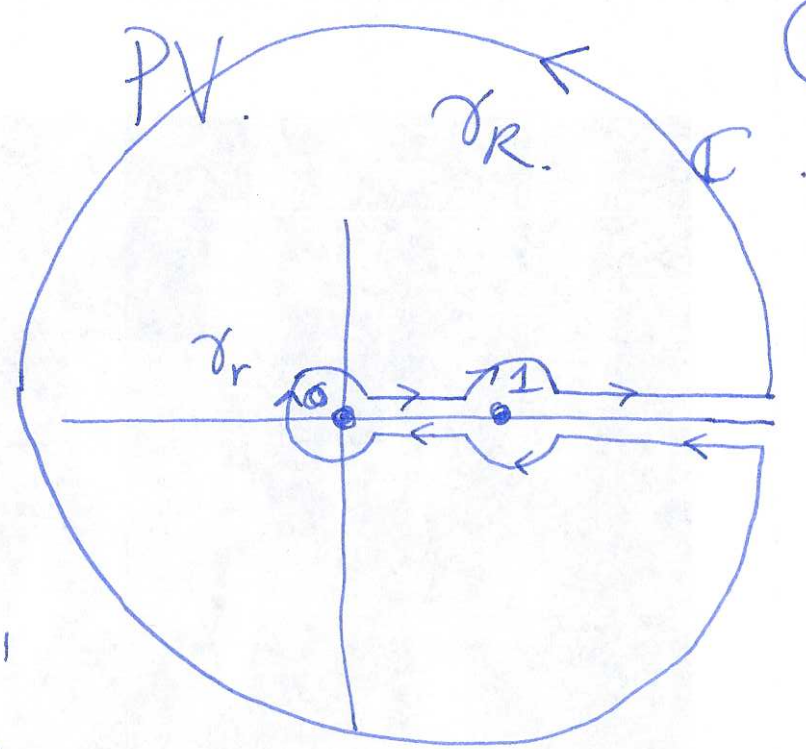
~~Wrong way~~ (Cauchy) PV

$$\lim_{t \to 0} \int_{-1}^{t} \frac{1}{x}\,dx + \int_{t}^{1} \frac{1}{x}\,dx$$

$$\lim_{t \to 0} \left[\ln|x|\right]_{-1}^{t} + \left[\ln|x|\right]_{t}^{1}.$$

$$\lim_{t \to 0} \underbrace{\cancel{\ln|t|}}_{\ln 2 + \cancel{\ln t}} - \underbrace{\ln(1)}_{0} + \underbrace{\ln(1)}_{0} - \ln\cancel{|t|}. = 0.$$

Q23 e) $\displaystyle\int_0^\infty \frac{x^{a-1}}{1-x}\,dx \qquad 0<a<1$



$$f(z) = \frac{e^{(a-1)\ln(z)}}{1-z}.$$

$\downarrow$

<u>intuition</u> $\quad z = x + 0i$

$$\frac{e^{(a-1)\ln(x)}}{1-x} = \frac{x^{a-1}}{1-x}.$$

$z = -x + 0i$

$$\frac{e^{(a-1)\ln(-x)}}{1-(-x)} \qquad \frac{e^{(a-1)\ln(x)+i\pi}}{1+x}. \qquad \frac{-x^{a-1}}{1+x}$$

<u>claim</u>. $\quad \left| \int_{\gamma_R} f(z)\,dz \right| \longrightarrow 0 \qquad \left| \int_{\gamma_r} f(z)\,dz \right| \longrightarrow 0$

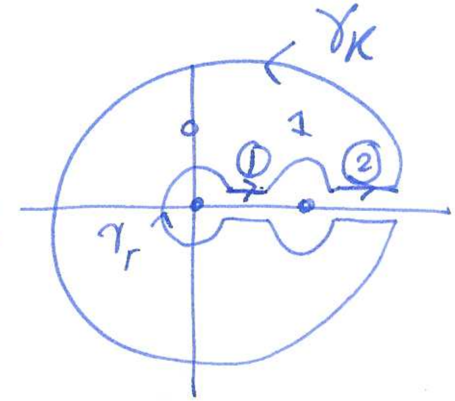
as $R \longrightarrow \infty$. $\qquad$ as $r \longrightarrow 0$.

$$f(z) = \frac{e^{(a-1)\ln(z)}}{1-z}.$$

$\ln(z)$

choose a branch

$0 < \arg \ln(z) < 2\pi$



1 2

$z(x) = \cancel{x+0i}$

$z(x) = x + \epsilon i.$

1 $$\int_r^{1-t} \frac{e^{(a-1)\ln(x+\epsilon i)}}{1-(x+\epsilon i)}\,dx$$

2 $$\int_{1-t}^{R} \frac{e^{(a-1)\ln(x+\epsilon i)}}{1-(x+\epsilon i)}\,dx.$$

$\lim$

$r \to 0$

$R \to \infty$

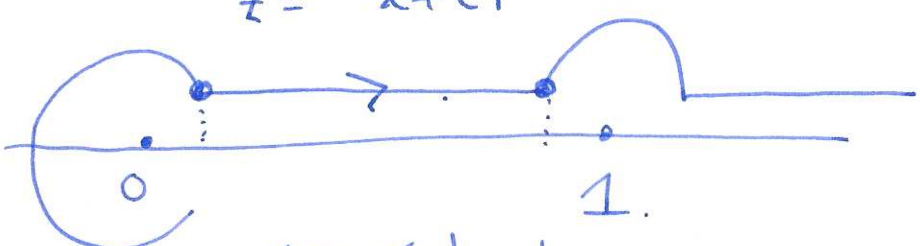
$\epsilon \to 0$

$t \to 0$.

1 $$\int_0^1 \frac{e^{(a-1)\ln(x)}}{1-x}\,dx + \int_1^\infty \frac{e^{(a-1)\ln(x)}}{1-x}\,dx = PV\int_0^\infty \frac{x^{a-1}}{1-x}\,dx$$

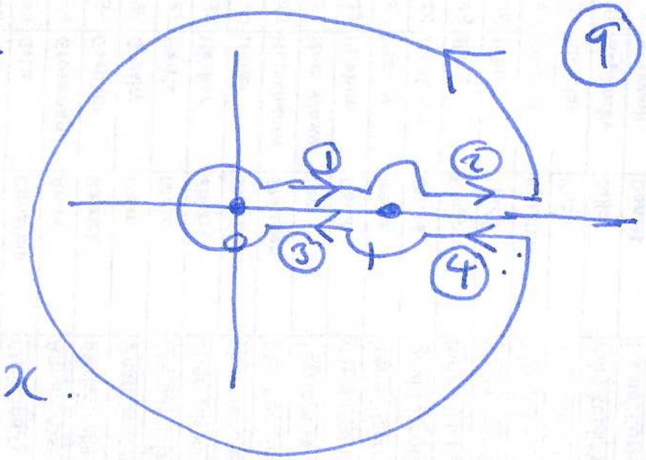
PV I.

$z = x + \epsilon i$



$r \le x \le 1-t.$

③, ④ $z(x) = x - i\epsilon$.



③ ④

$$\int_{1-t}^{r} \frac{e^{(a-1)\ln(x-i\epsilon)}}{1-(x-i\epsilon)}\,dx + \int_{R}^{1+t} \frac{e^{(a-1)\ln(x-i\epsilon)}}{1-(x-i\epsilon)}\,dx.$$

$$f(z) = \frac{e^{(a-1)\ln(z)}}{1-z}.$$

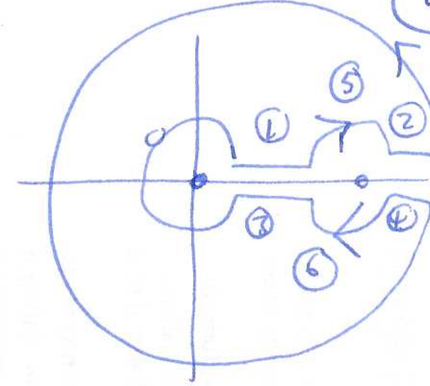
$$-\int_{r\to 0}^{1-t} \frac{e^{(a-1)\ln(x-i\epsilon)}}{1-(x-i\epsilon)}\,dx \;\overset{-}{\cancel{+}}\; \int_{1+t}^{R\to\infty} \frac{e^{(a-1)\ln(x-i\epsilon)}}{1-(x-i\epsilon)}\,dx.$$

$0 < \underset{\ln z}{\arg} < 2\pi$.

$\cancel{-PV\int_0^\infty e^{(a-1)\ln}}$ @ $\lim_{\epsilon\to 0} \ln(x-i\epsilon) = \ln(x) + 2\pi i$.

$$-PV\int_0^\infty \frac{e^{(a-1)(\ln x + 2\pi i)}}{1-x}\,dx = PV\int_0^\infty \frac{x^{a-1}}{1-x}\,dx \times e^{(a-1)2\pi i}$$

① + ② = I ③ + ④ = $-e^{(a-1)2\pi i}\, I$.



⑤ $z(\theta) = 1 + te^{i\theta}$ $\theta \in [\pi, 0]$.

$\frac{dz}{d\theta} = ite^{i\theta}$

$$\int_{\pi}^{0} \frac{e^{(a-1)\ln(1+te^{i\theta})}}{\cancel{1} - (\cancel{1}+te^{i\theta})}\, ite^{i\theta}\, d\theta.$$

$0 < \arg \ln z < 2\pi$

$$+\int_{0}^{\pi} \frac{e^{(a-1)\ln(1+te^{i\theta})}}{+\cancel{te^{i\theta}}}\, i\cancel{te^{i\theta}}\, d\theta = i\int_{0}^{\pi} e^{(a-1)\ln(1+te^{i\theta})}\, d\theta.$$

$$= \boxed{\pi i}$$ ⑤.

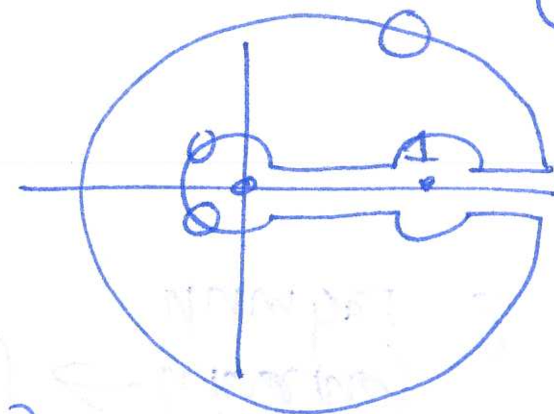
⑥ $z(\theta) = 1 + te^{i\theta}$ $\frac{dz}{d\theta} = ite^{i\theta}$ $\theta \in [2\pi, \pi]$.

$$\int_{2\pi}^{\pi} \frac{e^{(a-1)\ln(1+te^{i\theta})}}{\cancel{1} - (\cancel{1}+\cancel{te^{i\theta}})}\, i\cancel{te^{i\theta}}\, d\theta = +i\int_{\pi}^{2\pi} e^{(a-1)2\pi i}\, d\theta$$ ⑥

$$= \boxed{\pi i e^{(a-1)2\pi i}}$$

$$I \cancel{4} - I e^{(a-1)2\pi i} \quad 1\,2\,3\,4 \quad 4\,5$$

$$+ i\pi + i\pi e^{(a-1)2\pi i} \quad 5,\,6 \quad = 0$$



$$I\left(1 - e^{(a-1)2\pi i}\right) + i\pi\left(1 + e^{(a-1)2\pi i}\right) = 0$$

$$I = PV \int_0^\infty \frac{x^{a-1}}{1-x}\,dx$$

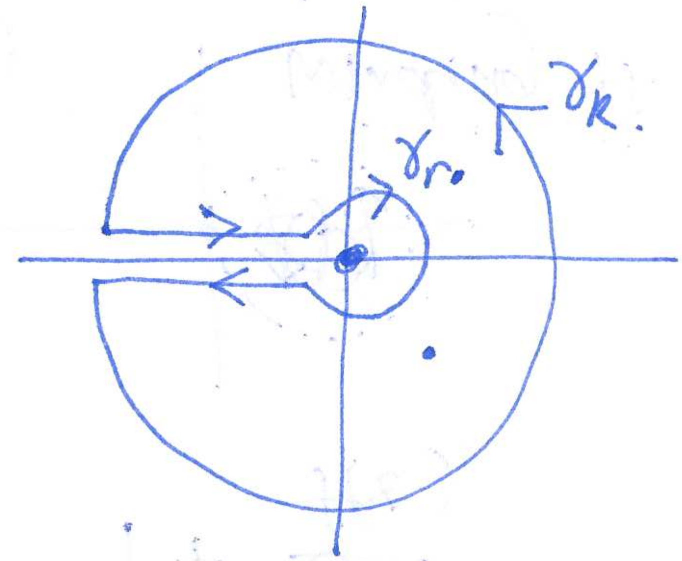
$$I = \frac{-\pi i\left(1 + e^{(a-1)2\pi i}\right)}{1 - e^{(a-1)2\pi i}} \frac{\times e^{-(a-1)\pi i}}{\times e^{-(a-1)\pi i}}$$

$$= -\pi i \frac{e^{-(a-1)\pi i} + e^{(a-1)\pi i}}{e^{-(a-1)\pi i} - e^{+(a-1)\pi i}} = -\pi i \frac{2\cos((a-1)\pi)}{\cancel{-}2i\sin((a-1)\pi)}.$$

$$\int_0^\infty \frac{x^{a-1}}{1-x}\,dx = +\pi i\,\frac{1}{i}\cot\left((a-1)\pi\right).$$

$$= +\pi\cot\left((a-1)\pi\right).$$

Q23 b) $\displaystyle\int_0^\infty \frac{\ln(x)}{(x+a)^2+b^2}\,dx \qquad \begin{matrix} a>0 \\ b>0 \end{matrix}$ $\qquad \mathbb{C}$: 18



$$f(z) = \cancel{\frac{\ln(z)}{(z+a)^2+b^2}}. \qquad \frac{\ln(z)^2}{(z-a)^2+b^2}.$$

poles at $a \pm ib$.

<u>claim</u> $\left| \int_{\gamma_R} \right| \to 0$ as $R \to \infty$.

$\left| \int_{\gamma_r} \right| \to 0$ as $r \to 0$.

$$z(x) = -x + i\epsilon \qquad +r < x < R$$

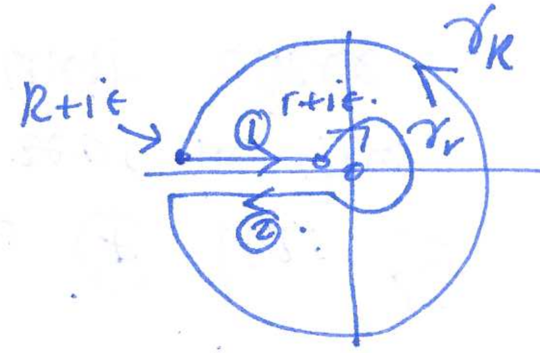
① $$\int_R^r \frac{\left(\ln(-x+i\epsilon)\right)^2}{(\overbrace{-x+i\epsilon}^{z}-a)^2+b^2}(-1)\,dx$$

$$\int_0^\infty \frac{\left(\ln(x)+\pi i\right)^2}{(x+a)^2+b^2}\,dx$$

② $$z(x) = -x - i\epsilon \qquad r < x < R$$

$$\int_r^R \frac{\left(\ln(-x+i\epsilon)\right)^2}{(-x-i\epsilon-a)^2+b^2}(-1)\,dx$$

$$-\int_0^\infty \frac{\left(\ln(x)-\pi i\right)^2}{(x+a)^2+b^2}\,dx.$$

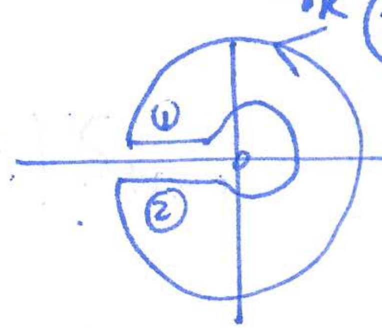


$$f(z) = \frac{\left(\ln(z)\right)^2}{(z-a)^2+b^2}.$$

$$-\pi < \arg \ln z < \pi$$

$>. > >.$

2d



1 + 2

$$\int_0^\infty \frac{(\ln(x)+\pi i)^2}{(x+a)^2+b^2}\,dx - \int_0^\infty \frac{(\ln(x)-\pi i)^2}{(x+a)^2+b^2}\,dx$$

$$\int_0^\infty \frac{\cancel{(\ln(x))^2} + 2\pi i \ln(x) - \cancel{(\pi i)^2} - \left[\cancel{(\ln(x))^2} - 2\pi i \ln(x) + \cancel{(\pi i)^2}\right]}{(x+a)^2+b^2}\cdot dx$$

$$\int_0^\infty \frac{4\pi i \ln(x)}{(x+a)^2+b^2}\,dx = 4\pi i \, \underline{I} = 2\pi i \sum \text{Res}.$$

$$f(z) = \frac{(\ln(z))^2}{(z-a)^2+b^2}.$$

poles order 1

$a+ib$

$a-ib$.

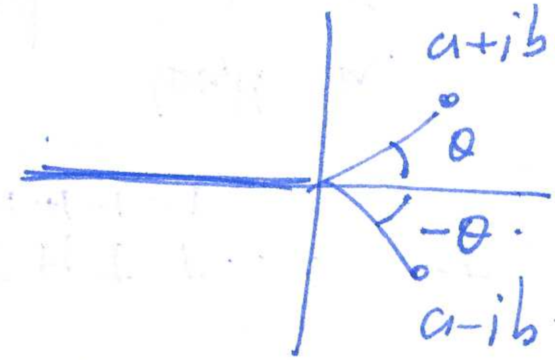
$$\operatorname*{Res}_{a+ib} f = \lim_{z \to a+ib} \frac{\cancel{(z-(a+ib))}\,(\ln(z))^2}{\cancel{(z-(a+ib))}\,(z-(a-ib))}$$

$$\cancel{\lim_{z \to a+ib}} \frac{\ln(a+ib)^2}{(a+ib-a+ib)} = \frac{\ln(a+ib)}{2ib}.$$

$$\operatorname*{Res}_{a-ib} f = \frac{\ln(a-ib)^2}{-2ib}.$$

$$\cancel{4\pi i}\, I = \frac{1}{2}\cancel{2\pi i}\left(\frac{\ln(a+ib)^2}{2ib} - \frac{\ln(a-ib)^2}{2ib}\right).$$

$$I = \frac{1}{4}\left(\frac{\ln(a+ib)^2}{\cancel{4}ib} - \frac{\ln(a-ib)^2}{ib}\right)$$



$\sqrt{a^2+b^2}$

$$I = \frac{1}{4ib}\Big[\big(\ln|a+ib| + i\arg(a+ib)\big)^2 - \big(\ln|a-ib| - i\arg(a-ib)\big)^2\Big]$$

$-\pi < \arg \ln(z) < \pi$

$\sqrt{a^2+b^2}$

$\tan^{-1}\left(\frac{b}{a}\right)$

$$I = \frac{1}{4ib}\Big[\cancel{(\ln\sqrt{a^2+b^2})^2} + \ln(\sqrt{a^2+b^2})\, i\arg(a+ib)$$

$$- \cancel{(\ln(\sqrt{a^2+b^2})^2} + \ln\sqrt{a^2+b^2}\, i\arg(a-ib)$$

$-\tan^{-1}\left(\frac{b}{a}\right)$

$$\cancel{(i\arg(a+ib))^2}$$

$$- \cancel{(i\arg(a-ib))^2}\Big]$$

(23)

$$I = \frac{1}{4ib} \ln \sqrt{a^2+b^2} \tan^{-1}\left(\frac{b}{a}\right) 2i.$$

$$I = \frac{1}{2b} \ln \sqrt{a^2+b^2} \tan^{-1}\left(\frac{b}{a}\right).$$

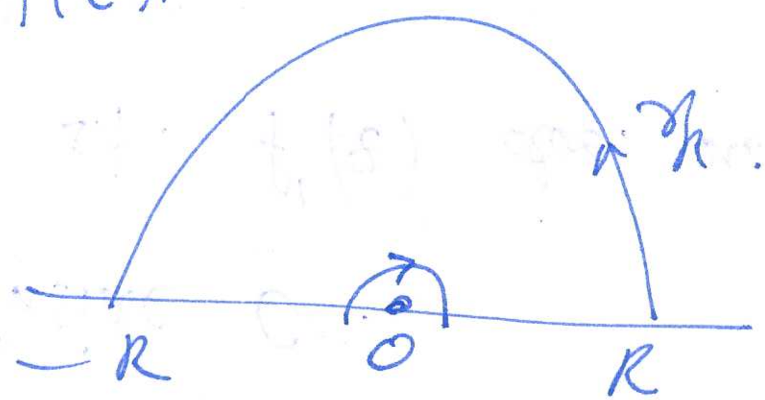
$$\int_0^\infty \frac{\ln x}{(x+a)^2+b^2}\,dx = \frac{\ln \sqrt{a^2+b^2} \tan^{-1}(b/a)}{2b}.$$

$(a-b)^2 \underline{-} (a+b)^2$ $\quad \cancel{a^2} - 2ab + \cancel{b^2}$

$\qquad\qquad\qquad\qquad -\cancel{a^2} - 2ab - \cancel{b^2} = -4ab.$

$$\int_{-\infty}^{\infty} f(x)\,dx$$

$f(z).$



$$\int_{0}^{\infty} f(x)\,dx.$$