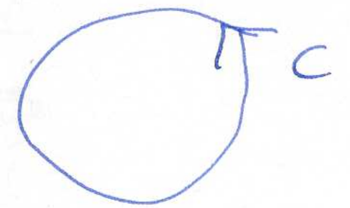
§12 Q4 a) $z^7 - \underbrace{5z^4}_{} + z^2 - 2$.



Rouché $f(z)$, $g(z)$ and

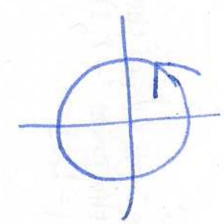
$|g(z)| < |f(z)|$ on C.

then $f(z)$ and $f(z)+g(z)$ have same number of zeros in C.

C

$f(z) = 5z^4$. $g(z) = z^7 + z^2 - 2$. $|z| = 1$

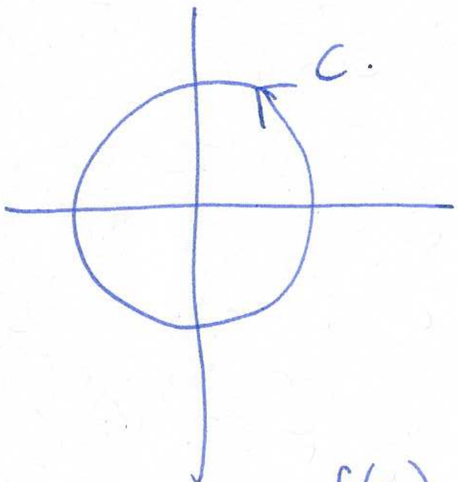
$|f(z)| = 5$ $|g(z)| \leq 1+1+2$



$4 < 5$.

$-f$ and $-f+g$ have same number of zeros

4 4



$f(z) \leftarrow$ # zeros inside C

(count w/ multiplicity).

e.g. $z^n \leftarrow$ 1 zero of order n.

n.

$z^n - \frac{1}{2} \leftarrow$ n zeros of order 1.

n.

$\underbrace{f(z) + g(z)}$

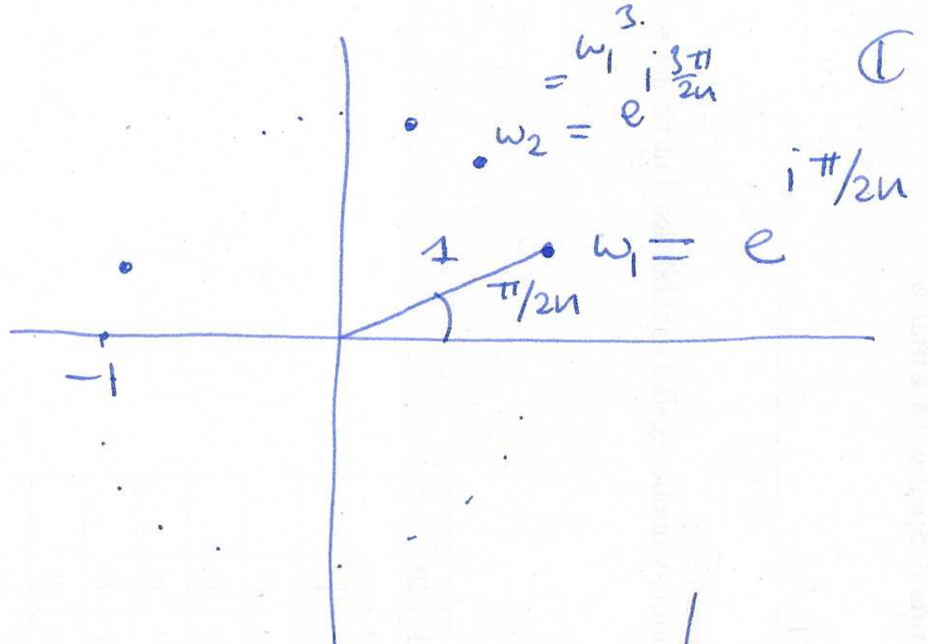
same number of zeros as f.

$|f(z)| > |g(z)|$.

§12 Q17

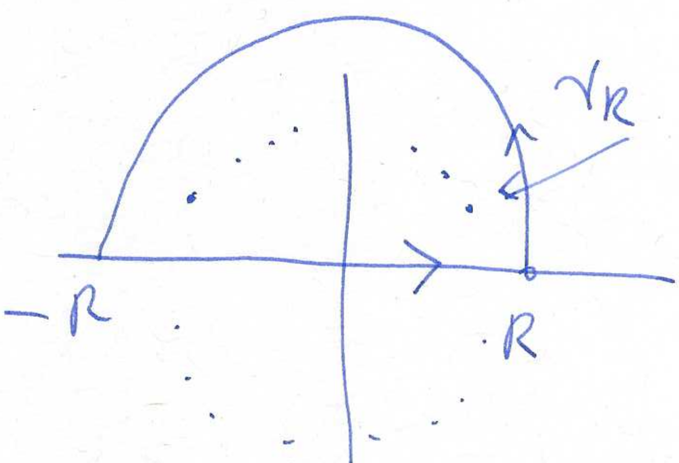
$$\int_0^\infty \frac{x^{2m}}{1+x^{2n}}\,dx. \qquad m<n. \qquad \frac{\pi}{n}\,\frac{1}{\sin\left(\frac{2m+1}{n}\pi\right)}.$$

$$f(z)=\frac{z^{2m}}{1+z^{2n}}.$$

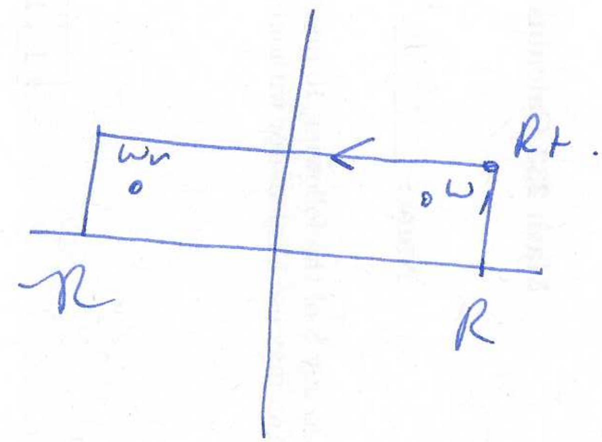


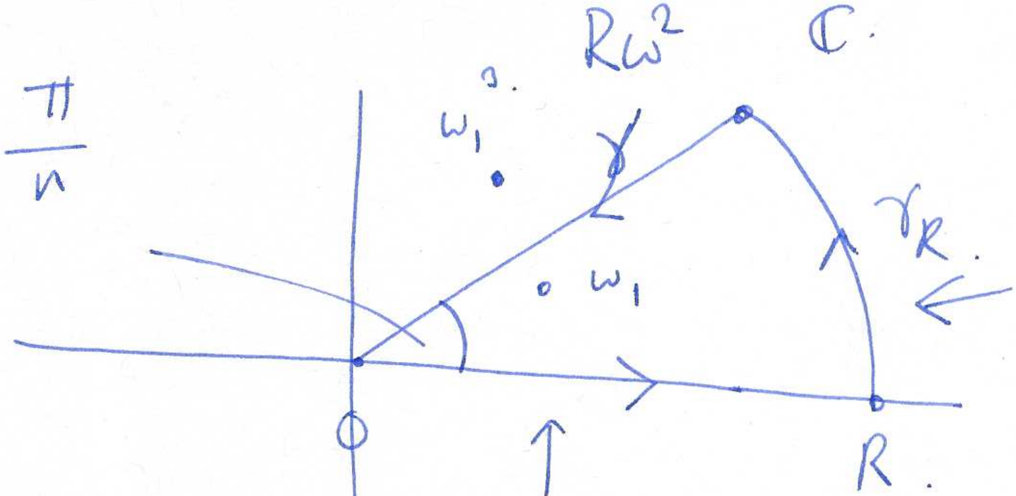
poles? $z^{2n}=-1$.

$2n$ poles on the unit circle ω_1^k. $k=1,3,\ldots,\cancel{2n+1}\ 4n-1$



n poles in here.





w $\cancel{w_1} = e^{i\pi/2n}$.

$$\underbrace{\int_{\gamma_R} \frac{z^{2m}}{1+z^{2n}}\,dz}$$

$$|\cdot| \le \frac{\pi R\, R^{2m}}{R^{2n}-1} \to 0$$ as $R \to \infty$, $m < n$.

$$\int_0^R \frac{x^{2m}}{1+x^{2n}}\,dx$$

$$\int_\gamma f(z)\,dz$$

parameterization $0 \le r \le R$

$\uparrow$

~~Bla~~ $rw^2 = re^{i\pi/n}$.

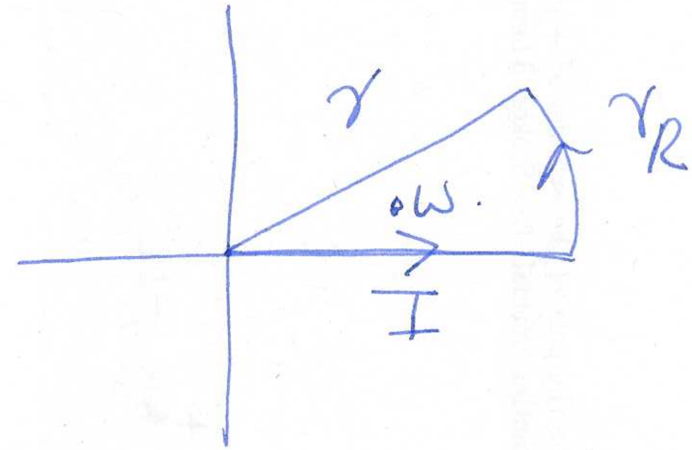
$z = rw^2$

$\frac{dz}{dr} = w^2$.

$$\int_R^0 \frac{(rw^2)^{2m}}{1+(rw^2)^{2n}} \underbrace{\frac{dz}{dr}}_{w^2}\, dr.$$

$$\omega = e^{i\pi/2n}$$

$$-\int_0^R \frac{r^{2m}\,\omega^{4m}}{1+r^{2n}\underbrace{\omega^{4n}}_{1.}}\,\omega^2\,dr.$$



$$-\omega^{4m+2}\underbrace{\int_0^R \frac{r^{2m}}{1+r^{2n}}\,dr}_{I.}$$

$$I-\omega^{4m+2}I+\underbrace{\int_{\gamma_R}}_{\to 0} = 2\pi i \operatorname*{Res}_{\omega} f(z).$$

$$I\left(1-\omega^{4m+2}\right) = 2\pi i \operatorname*{Res}_{\omega.} f(z).$$

$$\operatorname*{Res}_{w} f(z) = \lim_{z \to w} \frac{(z-w)\, z^{2m}}{1+z^{2n}}.$$

$$= \lim_{z \to w} \frac{z^{2m+1} - w z^{2m}}{1+z^{2n}}$$

l'H

$$= \lim_{z \to w} \frac{(2m+1)\, z^{2m} - w\, 2m\, z^{2m-1}}{2n\, z^{2n-1}}$$

$$= \frac{(\cancel{2m}+1)\, w^{2m} - \cancel{2m}\, w^{2m}}{2n\, w^{2n-1}} = \frac{w^{2m}}{2n\, w^{2n-1}}.$$

$$I\left(1 - w^{4m+2}\right) = \cancel{2}\pi i \, \frac{w^{2m}}{\cancel{2}n \, w^{2n-1}}. \qquad w = e^{i\pi/2n}$$

$$w^{2n} = -1.$$

$$= \frac{\pi i}{n} \cdot \frac{w^{2m}}{\cancel{} - 1/w} \qquad w^{2n-1} \cdot w = -1.$$

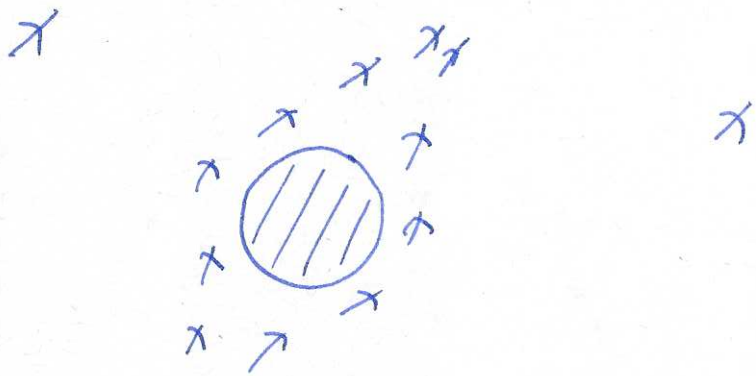
$$= \frac{\pi i}{n} \, w^{2m+1}.$$

$$I = \frac{\pi i}{n} \, \frac{w^{2m+1}}{1 - w^{4m+2}} = \frac{\pi i}{n} \, \frac{1}{w^{-2m-1} - w^{2m+1}}.$$

$$I = \frac{2\pi}{n} \sin\left(\frac{2m+1}{n}\pi\right).$$

$$e^{i\frac{\pi}{2n}(2m+1)}.$$

flow past a cylinder / disc radius R



- assume velocity at ∞ is $w_\infty = u_\infty + i v_\infty$.

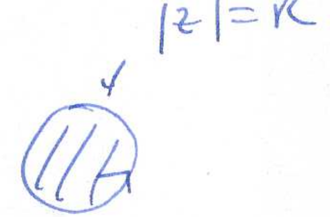
- assume flow solenoidal irrotational.

$f(z)$ complex potential for the flow

$f'(z)$ analytic in $|z| > R$

$f'(\infty) = \overline{w}_\infty$.

$f(z)$ complex potential analytic in $|z| > R$



so $f'(z)$ has a Laurent expansion

$$f'(z) = \bar{w}_\infty + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots$$

$$f(z) = \bar{w}_\infty z + c_1 \ln(z) - \frac{c_2}{z} - \frac{c_3}{2z^2} + \cdots$$

stream/flow function $\psi(r, \theta) = \operatorname{Im} f(z)$ $\quad z = re^{i\theta}$

set $c_1 = a_1 + b_1 \cancel{z} i$

$c_2 = a_2 + b_2 i$

$\vdots$

$$\psi(r,\theta) = \operatorname{Im} f(z) = \operatorname{Im}\Big(\underbrace{\bar{w}_{\infty}}_{1} z + \underbrace{c_1}_{a_1+b_1 i} \ln(z) - \frac{\overbrace{c_2}^{a_2+b_2 i}}{z} - \dots \Big).$$

assume ~~tkdek~~.

$$w_\infty = u_\infty + i v_\infty.$$

$\cancel{\psi(r,\theta)} =$

$$z = re^{i\theta}.$$

$$z = r\cos\theta + i r\sin\theta$$

$$\ln(z) = r + i\theta$$

$$\frac{c_2}{z} = \frac{(a_2 + i b_2)\cancel{z} e^{-i\theta}}{r}$$

$$\psi(r,\theta) = u_\infty r\cos\theta - v_\infty r\sin\theta$$

$$+ a_1 r - b_1 \theta$$

$$+ a_2$$

$$\psi(r,\theta) = \operatorname{Im} f^{\delta}(z) = \operatorname{Im}\left(\underset{u_\infty + i v_\infty}{\bar{w}_\infty} z + \underset{a_1+ib_1}{c_1} \ln(z) - \underset{a_2+ib_2}{\frac{c_2}{z}} - \frac{c_3}{2z^2} \cdots \right)$$

$\cdots$

$$z = re^{i\theta} = r\cos\theta + i\sin\theta$$

$$\ln(z) = \underset{\ln(r)}{\delta} + i\theta$$

$$\psi(r,\theta) = u_\infty r\sin\theta + v_\infty r\cos\theta$$
$$+ a_1\theta + b_1 \ln(r)$$
$$+ \frac{a_2 \sin\theta}{r} - \frac{b_2\cos\theta}{r}$$
$$+ \frac{a_3\sin 2\theta}{2r^2} - \frac{b_3\cos 2\theta}{2r^2}$$
$$+ \cdots$$

$$-\frac{c_2}{z} = -\frac{a_2 - ib_2}{r} e^{-i\theta}$$
$$-\frac{a_2+ib_2}{r}(\cos\theta - i\sin\theta)$$

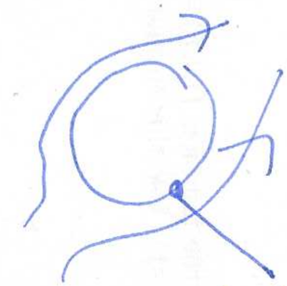
$$-\frac{c_3}{2z^2} = -\frac{c_3}{2}\frac{e^{-2i\theta}}{r^2}$$

$$\psi(r,\theta) = a_1\theta + b_1 \ln(r) + \frac{a_2 + r^2 u_\infty}{r}\sin\theta \cancel{+} - \frac{b_2 + r^2 v_\infty}{r}\cos\theta$$

$$+ \frac{a_3}{2r^2}\sin 2\theta \cancel{+} - \frac{b_3}{2r^2}\cos\theta + \cdots$$

Let Γ be the circle $|z| = R$ ← this must be a flow line

$$\psi(R,\theta) = \text{const}$$



|z|=R.

one solutions:

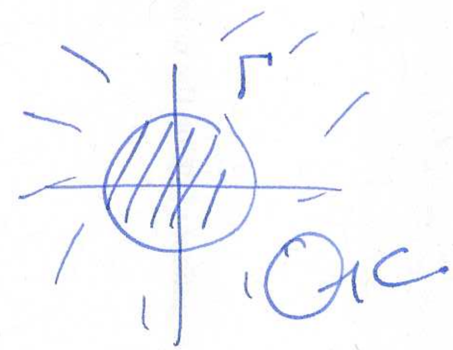
$$a_1 = 0$$

$$a_2 = -R^2 u_\infty$$

$$b_2 = -R^2 v_\infty$$

$$a_3 = b_3 = \cdots = 0.$$

$$\psi(r,\theta) = b_1 \ln(r) + \frac{-R^2 u_\infty + r^2 u_\infty}{r} \sin\theta - \frac{-R^2 v_\infty + r^2 v_\infty}{r} \cos\theta .$$



gives

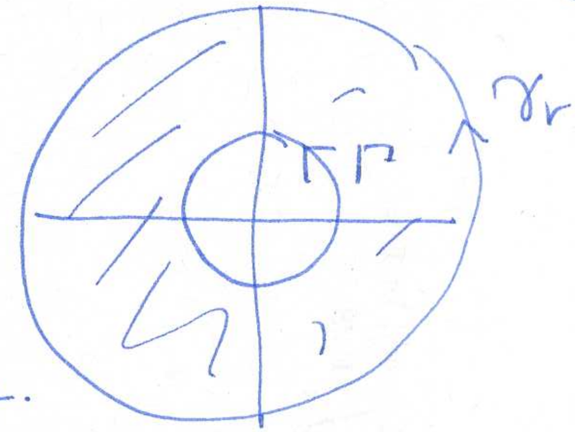
$$f'(z) = \bar{w}_\infty + \frac{i b_1}{z} - \frac{R^2 w_\infty}{z} .$$

$$f(z) = i b_1 \ln(z) + \bar{w}_\infty z + \frac{R^2 w_\infty}{z} .$$

say K circulation around Γ.

$$K = \mathrm{Re} \int_{\partial\Gamma} f'(z)\, dz = \mathrm{Re} \int_{|z|=R} f'(z)\, dz .$$

$$= \operatorname{Re} \int_{|z|=r} f'(z)\,dz \quad \text{for any } r > R.$$

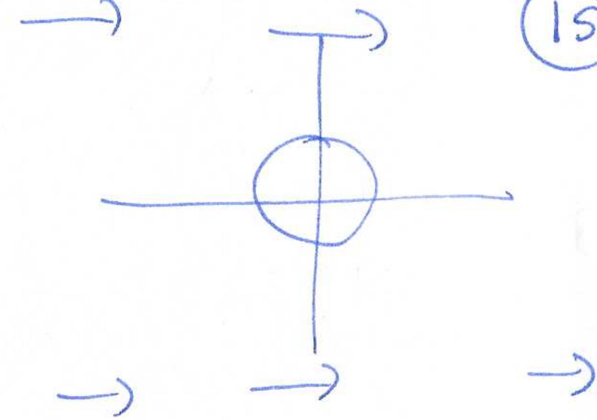


$$\kappa = \operatorname{Re} \int_{|z|=r} \left\{ \bar{w}_\infty + \frac{i b_1}{z} + \frac{R^2 w_\infty}{z^2} \right\} dz = -2\pi b_1.$$

so $b_1 = -\dfrac{\kappa}{2\pi}$.

$$f'(z) = \bar{w}_\infty + \frac{\kappa}{2\pi i\, z} - \frac{R^2 w_\infty}{z^2}.$$

$$f(z) = \frac{\kappa}{2\pi i} \ln(z) + \bar{w}_\infty z + \frac{R^2 w_\infty}{z}.$$



wlog $\bar{w}_\infty \in \mathbb{R}_{>0}$.

$v_\infty = 0 \quad u_\infty \in \mathbb{R}_{>0}$.

find the stagnation points $f'(z) = 0$

$$f'(z) = \bar{w}_\infty + \frac{K}{2\pi i z} - \frac{R^2 w_\infty}{z^2} = 0$$

quadratic

$$z^2 + \frac{K}{2\pi i w_\infty} z - R^2 = 0$$

solutions

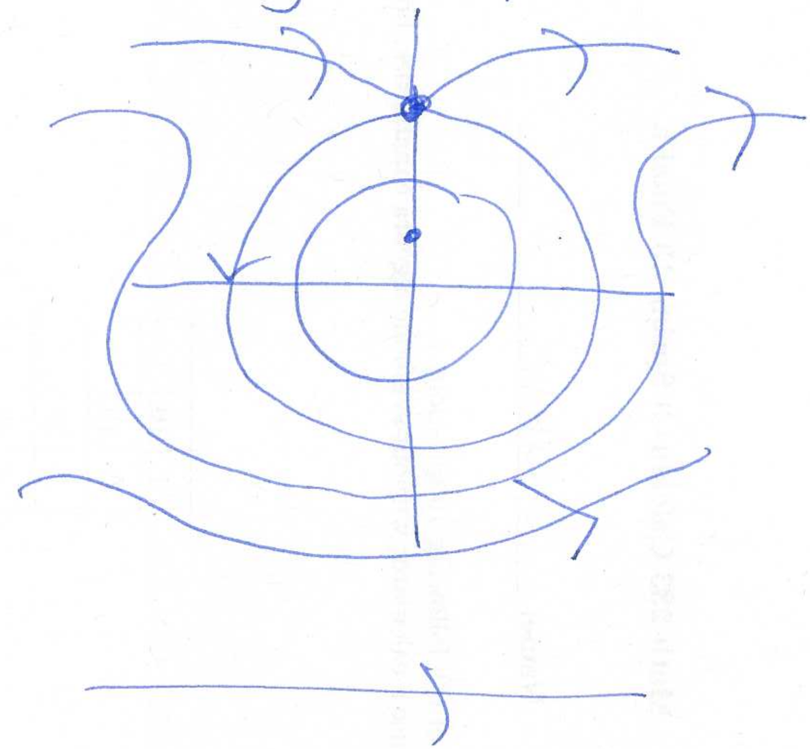
$$z_1, z_2 = \frac{iK}{4\pi u_\infty} \pm \sqrt{R^2 - \left(\frac{K}{4\pi u_\infty}\right)^2}$$

$$z_1, z_2 = \frac{i\kappa}{4\pi u_\infty} \pm \sqrt{R^2 - \left(\frac{\kappa}{4\pi u_\infty}\right)^2}.$$

- $|\kappa| > 4\pi R u_\infty$ $z_1 z_2$ purely imaginary, but $z_1 z_2 = R^2$.

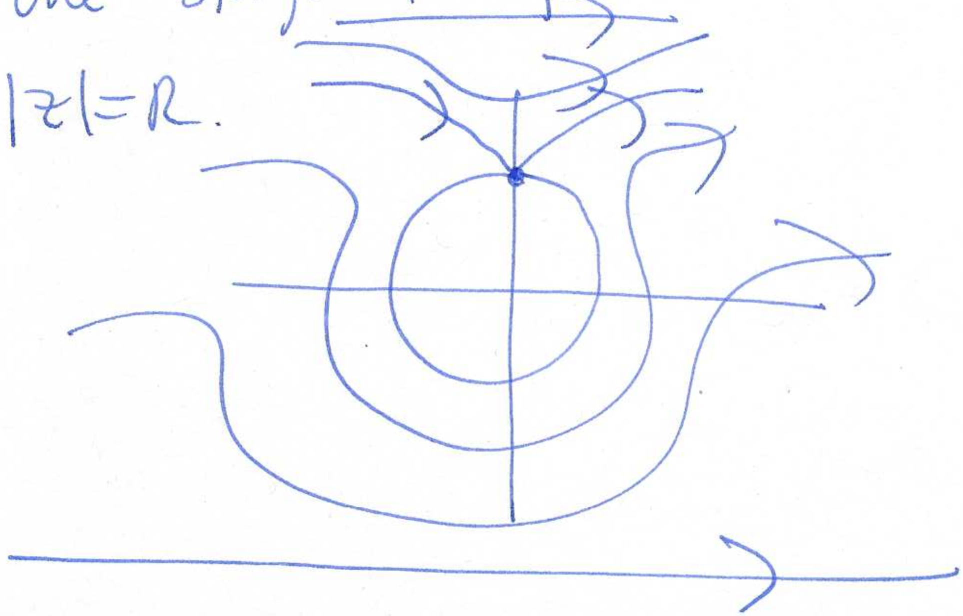
only one outside $|z| = R$

get

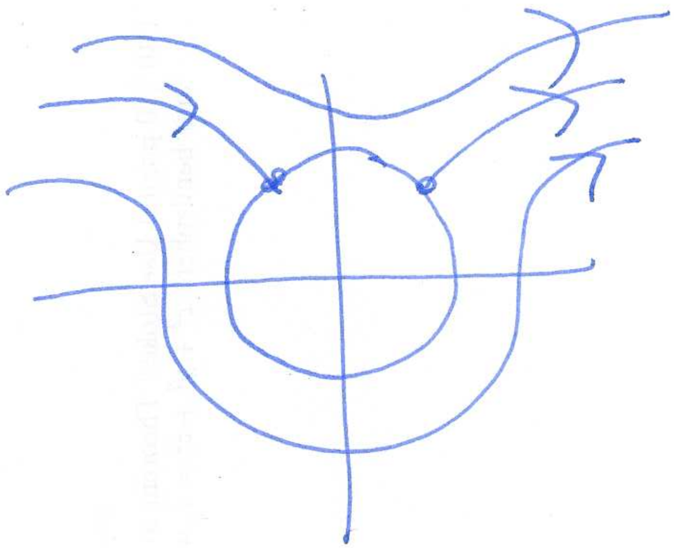


- $|\kappa| = 4\pi R_\infty$

only one stagnation pt on $|z| = R$.

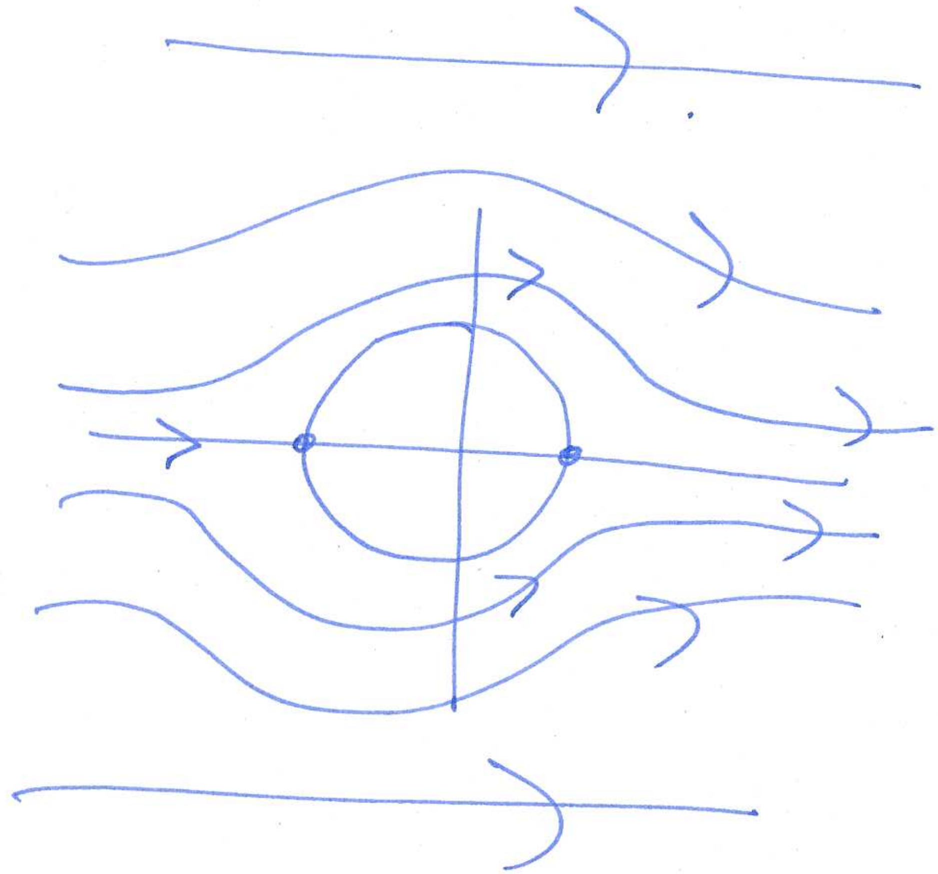


• $|K| < 4\pi R u_\infty$ two stagnation points (both on $|z| = R$).

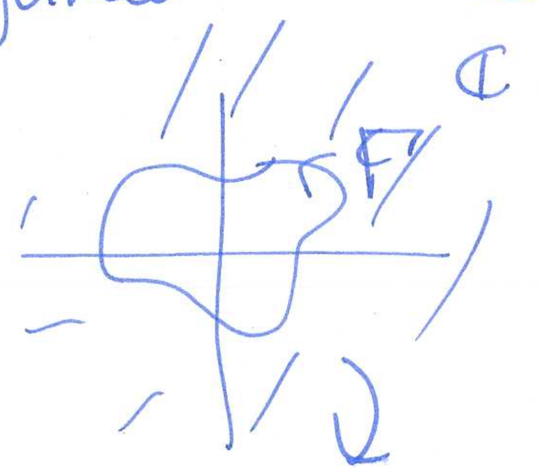


• $|K| = 0$

(no circulation)

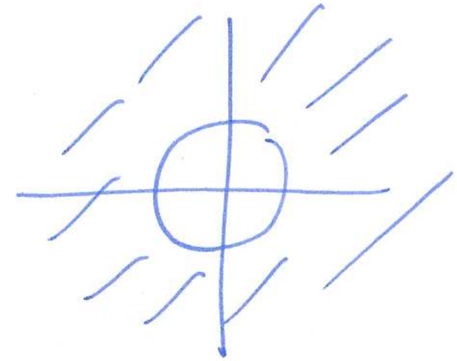


Q: spose obstacle is not a circle/cylinder.



Fact: there is a unique conformal map that takes

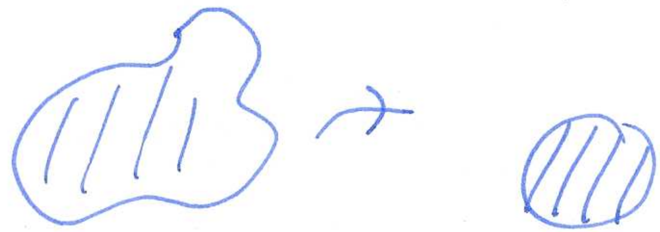
$\mathbb{C} \setminus \mathrm{int}(\Gamma)$ to $\mathbb{C} \setminus \mathrm{int}\ |z| \leq 1$.



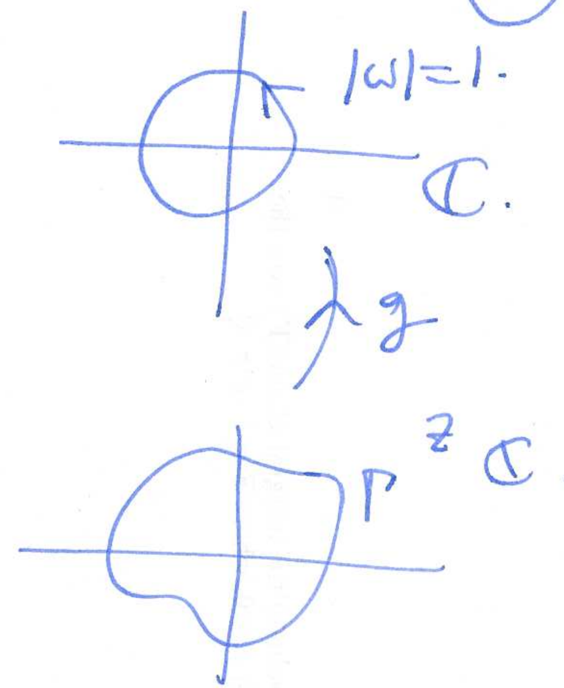
s.t. $g(\infty) = \infty$

$g'(\infty) \in \mathbb{R}_{>0}$.

Laurent expansion of g



$$g(z) = cz + c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots$$



solution for $|w|=1$

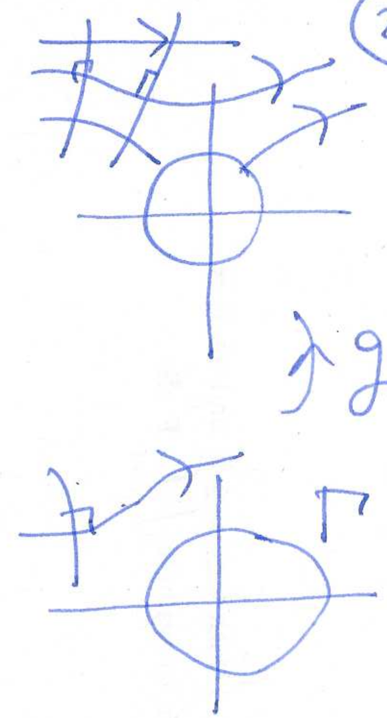
is $\Phi(w) = \frac{K}{2\pi i} \ln(w) + \bar{A} w + \frac{A}{w}.$

So $f(z) = \cancel{\frac{K}{2\pi i} \ln(g(z)) + \frac{\bar{w}_\infty}{g'(\infty)}}$

$$f(z) = \frac{K}{2\pi i} \ln(g(z)) + \bar{A} g(z) + \frac{A}{g(z)}$$

$$\bar{w}_\infty = f'(\infty) = \Phi'(\infty)\, g'(\infty) = \bar{A}\, c$$

$$A = \frac{w_\infty}{c} = \frac{w_\infty}{g'(\infty)}.$$



so

$$f(z) = \frac{k}{2\pi i}\ln(g(z)) + \frac{\bar{w}_\infty}{g'(\infty)}\, g(z) + \frac{w_\infty}{g'(\infty)\, g(z)}.$$

can find flow lines if you know $g(z)$.

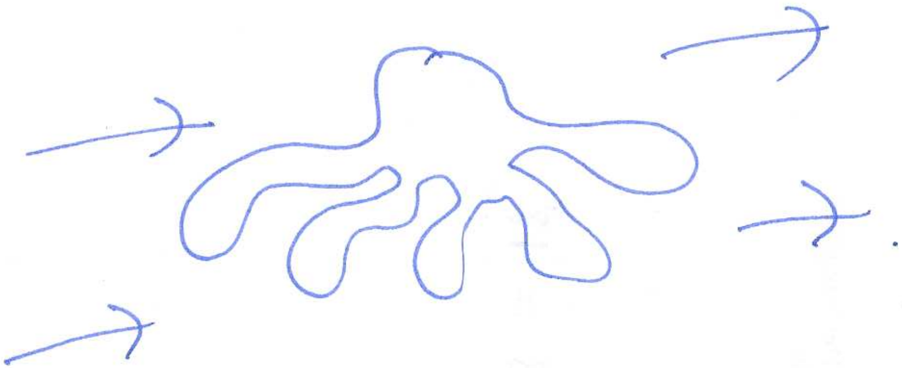
<u>Application</u> (Kutta-Joukowski Thm).

can find force exerted on the plane wing w/ cross section Γ by flow w/ $w_\infty = u_\infty + i v_\infty$

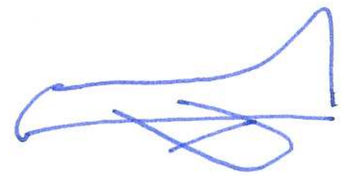
<u>A</u>: $\boxed{F = -\rho k \underline{w_\infty}\, i}$

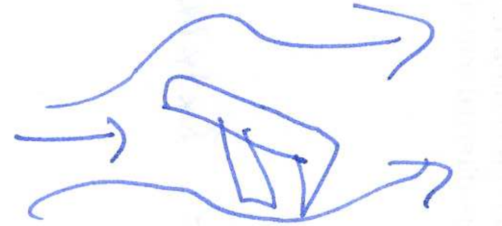
k = circulation

ρ = density of fluid = constant



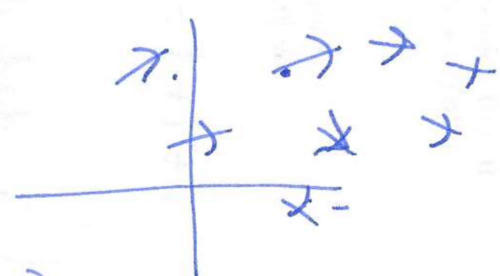
$$F = -\rho \kappa w_\infty i.$$





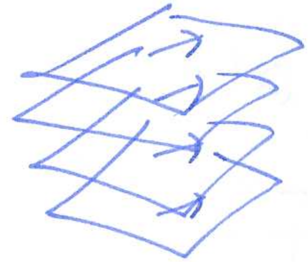
§15.3 Electrostatics

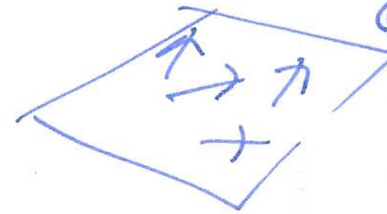
electric field $\leftrightarrow$ vector field: gives the force experienced by a unit positive.



<u>assumptions</u> · stationary (don't depend on time).

plane-parallel / 2d.





$\mathbb{C}$.

— same in all parallel planes

— has no component $\perp$ to the plane

$$E(x,y) = E_x(x,y) + i\,E_y(x,y).$$

<u>Maxwell's equation</u>

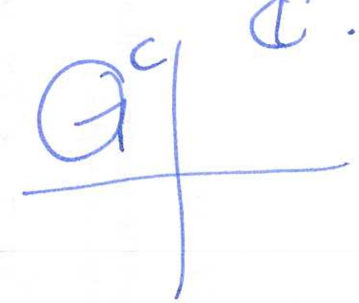
$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 4\pi \rho.$$

charge density

consider G simply connected domain
with $\rho = 0$ in G

Green's Thm



$$\int_C E_x\,dx + E_y\,dy = 0 \qquad \int_C -E_y\,dx + E_x\,dy = 0$$

so the electrostatic field is
irrotational   solenoidal.

so both $E_x\,dx + E_y\,dy$ and $-E_y\,dx + E_x\,dy$ are exact

so their are real functions. $\phi(x,y)$ $\psi(x,y)$.

$$-E_x\,dx - E_y\,dy = d\phi \qquad E_x = -\frac{\partial \phi}{\partial x} \qquad E_y = -\frac{\partial \phi}{\partial y}$$

$$-E_y\,dx + E_x\,dy = d\psi \qquad E_y = -\frac{\partial \psi}{\partial x} \qquad E_x = \frac{\partial \psi}{\partial y}$$

so $\dfrac{\partial \psi}{\partial x} = \dfrac{\partial \phi}{\partial y}$ $\qquad \dfrac{\partial \psi}{\partial y} = -\dfrac{\partial \phi}{\partial x}$ $\leftarrow$ CR equations.

$f(z) = \psi + i\phi$ analytic in $\mathbb{C}$

$$f'(z) = \frac{\partial \psi}{\partial x} + i\frac{\partial \phi}{\partial x} = -E_y - iE_x = -i(E_x - iE_y).$$

equivalently $E = E_x + iE_y = -i\,\overline{f'(z)}$

ψ stream function $\qquad \psi(x,y) = \text{const}$ lines of force

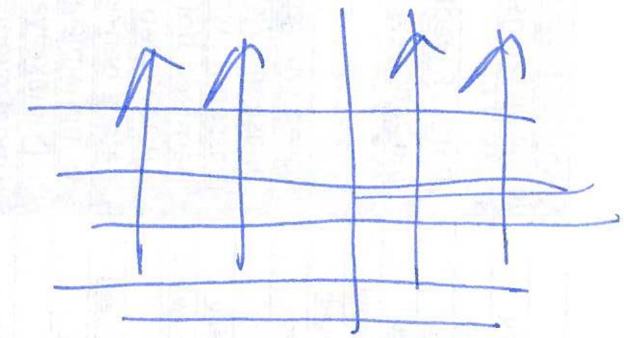
ϕ potential $\qquad \phi(x,y) = \text{const}$ equipotentials.

Γ conducting surface $\leftrightarrow$ part of an equipotential

i.e. electric field can have no component tangent to Γ, this would induce a current / motion of charge in Γ.

$\not\#$ stationarity

$\mathbb{C}$



<u>Examples</u> $f(z) = az$ $\quad a \in \mathbb{R} > 0$

complex potential

electric field $E_x + iE_y = -if'(z) = -ia.$

$E_x = 0 \quad E_y = -a.$

$f(z) = ax + iay$

$\psi = ax = \text{const}$ lines of force

$\phi = ay = \text{const}$ equipotentials