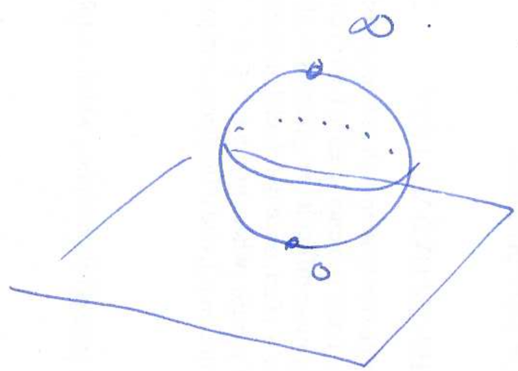
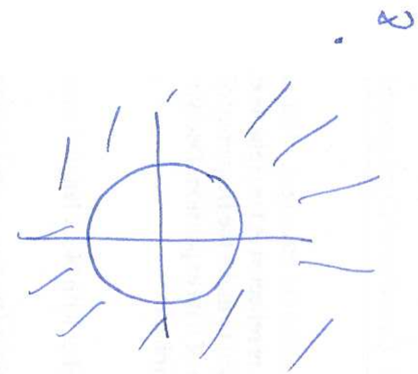
$$f(z) \qquad \cdots + \frac{c_{-2}}{z^2} + \frac{c_{-1}}{z} + c_0 + c_1 z + c_2 z^2 + \cdots$$

$$g(z) = f\left(\frac{1}{z}\right)$$



$|z| \geq R.$



$$g(z) = \cdots + c_{-2} z^2 + \cancel{\frac{c_{-1}}{z}}\, c_{-1} z + c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots$$

$\left|\frac{1}{z}\right| \geq R.$

$|z| \leq R.$

Q11 c) g).

$z=0$.

c) $f(z) = \dfrac{1}{\sin(\frac{1}{z})}$

$\sin(\frac{1}{z}) = 0$

$\frac{1}{z} = n\pi$. $z = \frac{1}{n\pi}$.

$z = \frac{1}{n\pi}$ simple poles.

$z = 0 \leftarrow$ not isolated singularity

$$\lim_{z \to n\pi} \frac{z - n\pi}{\sin(\frac{1}{z})}$$

g) $e^{\cot\left(\frac{1}{z}\right)}$.

$\cot\left(\frac{1}{z}\right)$ ← pole.

$= \dfrac{\cos\left(\frac{1}{z}\right)}{\sin\left(\frac{1}{z}\right)}$

$\frac{1}{z} = \pi n$.

$z = \frac{1}{\pi n}$ ~~poles~~.

$z = 0$ not isolated singularity.

$\frac{1}{z}$ ← pole at 0.

$e^{1/z}$ ← essential sing at 0.

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$$ ← essential sing

$\cot\left(\frac{1}{z}\right)$ has a pole at $\frac{1}{n\pi}$.

$$\cot\left(\frac{1}{z}\right) = \frac{c_{-1}}{z - \frac{1}{n\pi}} + c_0 + c_1\left(z - \frac{1}{n\pi}\right) + c_2\left(z - \frac{1}{n\pi}\right)^2 + \cdots$$

$$e^{\cot\left(\frac{1}{z}\right)} = e^{\frac{c_{-1}}{z - \frac{1}{n\pi}} + c_0 + c_1\left(z - \frac{1}{n\pi}\right) + \cdots}$$

$$e^{\frac{c_{-1}}{z - 1/n\pi}} \underbrace{e^{c_0 + c_1\left(z - \frac{1}{n\pi}\right) + \cdots}}$$

$$\left(1 + \frac{c_{-1}}{z - \frac{1}{n\pi}} + \frac{c_{-1}}{2!\left(z - \frac{1}{n\pi}\right)^2} + \cdots\right)\left(a_0 + a_1 \cancel{z}\left(z - \frac{1}{n\pi}\right) + a_2 \cancel{z^2}\left(z - \frac{1}{n\pi}\right)^2 + \cdots\right).$$

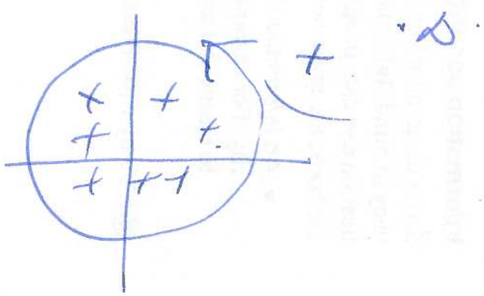
<u>Useful fact</u> it makes sense to define $\operatorname{Res}_{\infty} f(z)$

<u>Defn</u>

$$\operatorname*{Res}_{z=\infty} f(z) = -\operatorname{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right)\right).$$

<u>Consequence</u> $f(z)$ has only poles ~~the~~ $z_1, \dots, z_n, \infty$.

then $\operatorname*{Res}_{z_1} f + \operatorname*{Res}_{z_2} + \dots + \operatorname*{Res}_{z_n} + \operatorname*{Res}_{\infty} = 0$.



"change of variable" $z \mapsto \frac{1}{z}$

$$dz \mapsto \frac{d}{dz}\left(\frac{1}{z}\right) = -\frac{1}{z^2}.$$

Group actions $\leftrightarrow$ isometries / similarities / Möbius maps.

$\mathrm{Isom}(\mathbb{R}^2) \curvearrowright \mathbb{R}^2 \leftrightarrow \mathbb{C}$.

$\uparrow$

$$\left\{ \begin{array}{l} z \mapsto e^{i\theta} z + a \\ z \mapsto e^{i\theta} \bar{z} + a \end{array} \;\middle|\; \begin{array}{l} \theta \in [0, 2\pi) \\ a \in \mathbb{C} \end{array} \right\}.$$

Q : what sort of isometries are there?

A : translations, rotations, reflections, glide reflections.

means : any isometry is conjugate to one of these.

conjugation : $g \in \mathrm{Isom}(\mathbb{R}^2)$ $\quad fgf^{-1} \leftarrow$ conjugate of g by f.

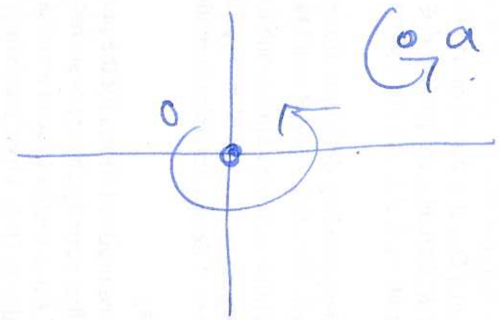
$f \in \mathrm{Isom}(\mathbb{R}^2)$

subgroup of $\mathrm{Isom}(\mathbb{R}^2)$ $\quad fHf^{-1} = \{ fhf^{-1} \mid h \in H \}$.

H.

Examples. (1) $H_0 = \{\text{rotation about } 0\} \leq \mathrm{Isom}(\mathbb{R}^2)$.

$\{ z \mapsto e^{i\theta} z \}$.



$H_a = \{\text{rotations about } a \in \mathbb{C}\}$.

$f : z \mapsto z + a$.

$H_a = f H_0 f^{-1}$

(2) Stab

$\mathrm{Isom}(\mathbb{R}^2) \curvearrowright \mathbb{R}^2 \leftrightarrow \mathbb{C}$.

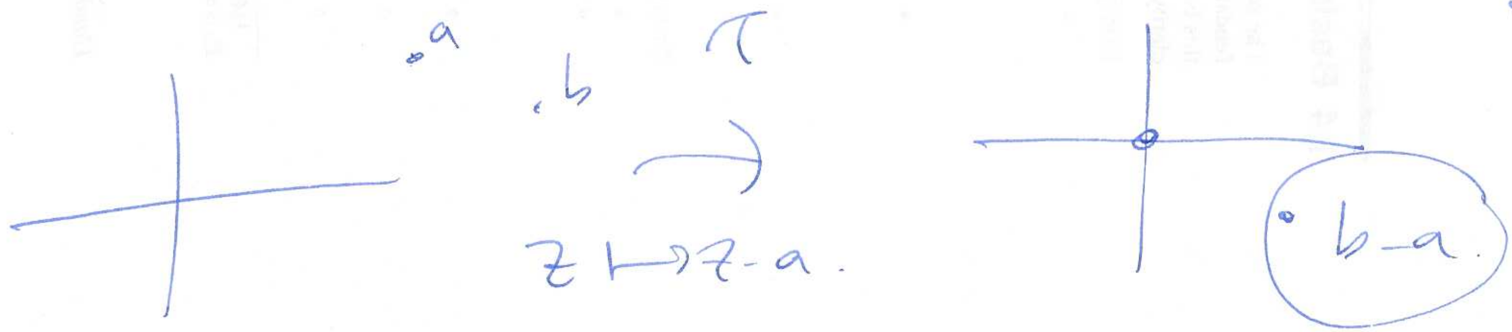
$$\mathrm{stab}(0) = \{ g \in \mathrm{Isom}(\mathbb{R}^2) \mid g(0) = 0 \}.$$

$$H_0 \subsetneq \mathrm{stab}(0) = \left\{ \begin{array}{l} z \mapsto e^{i\theta} z \\ z \mapsto e^{i\theta} \bar{z} \end{array} \right\}.$$

<u>check</u> $z \mapsto e^{i\theta} z + a$. fixes 0: $f(0) = 0 = a$.

$z \mapsto e^{i\theta} \bar{z} + a$ fixes 0: $f(0) = 0 = a$.

unordered pairs of points $\{a, b\} \to \{0, b-a\}$.



orbit under $\mathrm{stab}(0)$.

$\mathrm{Isom}(\mathbb{R}^2)$ claim:

every element of Isom is conjugate to one of these

$$\begin{cases} \text{rotation} & \{z \mapsto e^{i\theta} z\}. \\ \text{translation} & \{z \mapsto z+a\}. \\ \text{reflection} & \{z \mapsto \bar{z}\}. \\ \text{glide reflection} & \{z \mapsto \bar{z} + r\} \quad r \in \mathbb{R}. \end{cases}$$

Proof (sketch). $z \mapsto e^{i\theta} z + a$.

special case $\theta = 0$: $z \mapsto z + a$. (translation) ✓.

suppose $\theta \neq 0$ $z \mapsto e^{i\theta} z + a$.

fixed point: $z = e^{i\theta} z + a$.

$z - e^{i\theta} z = a$ $\qquad z = \dfrac{a}{1 - e^{i\theta}}$.

$f(z) = e^{i\theta} z + a.$ conjugate with $z \overset{g}{\longmapsto} z + \frac{a}{1-e^{i\theta}}$.

$g^{-1} f g$

$$z \overset{g}{\longmapsto} z + \frac{a}{1-e^{i\theta}} \overset{f}{\longmapsto} e^{i\theta}\left(z + \frac{a}{1-e^{i\theta}}\right) + a \overset{g^{-1}}{\longmapsto}$$

$$e^{i\theta}\left(z + \frac{a}{1-e^{i\theta}}\right) + a - \frac{a}{1-e^{i\theta}} = e^{i\theta} z + \frac{a}{1-e^{i\theta}}\left(e^{i\theta} - 1\right) + a.$$

$$= e^{i\theta} z - \cancel{a} + \cancel{a} = e^{i\theta} \leftarrow \text{rotation } \checkmark.$$

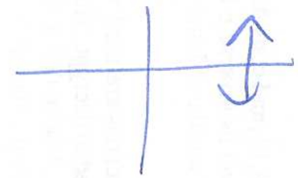
$e^{i\theta} z + a \leftarrow$ either a translation $(\theta = 0)$

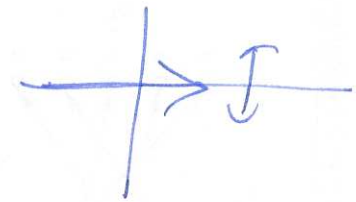
<u>or</u> conjugate to a rotation about 0.

$f(z) = e^{i\theta}\bar{z} + a.$ reflections $z \mapsto \bar{z}$

glide reflection $z \mapsto \bar{z} + r$, $r \in \mathbb{R}$.

conjugate by a rotation $\overset{\rho_\phi}{e^{i\phi}}$.





$\rho_\phi f \rho_\phi^{-1}$

$$z \xmapsto{\rho_\phi^{-1}} e^{-i\phi} z \xmapsto{f} e^{i\theta}\overline{\left(e^{-i\phi} z\right)} + a = e^{i(\theta+\phi)}\bar{z} + a$$

$$\xmapsto{\rho_\phi} e^{i\phi}\left(e^{i(\theta+\phi)}\bar{z} + a\right) = e^{i(\theta+2\phi)}\bar{z} + \underbrace{e^{i\phi}a}_{\in \mathbb{C}}.$$

choose $\phi = -\frac{\theta}{2}$.

this conjugates $f(z) = e^{i\theta}\bar{z} + a$

to $f'(z) = \bar{z} + \underbrace{e^{i\phi}a}_{\in \mathbb{C}}.$

$f(z) = \bar{z} + a.$ $\qquad a = x + iy$

conjugate by $z \mapsto z + bi$ τ. $b \in \mathbb{R}$.

$\tau f \tau^{-1}: \quad z \xmapsto{\tau^{-1}} z - bi \xmapsto{f} \overline{z - bi} + a \xmapsto{\tau}$

$\bar{z} + bi + a + bi = \bar{z} + \underset{\substack{\| \\ x+iy}}{a} + 2bi$ $\qquad$ choose $b = -\frac{y}{2}$

$\bar{z} + a$ is conjugate to $\bar{z} + x \in \mathbb{R}$

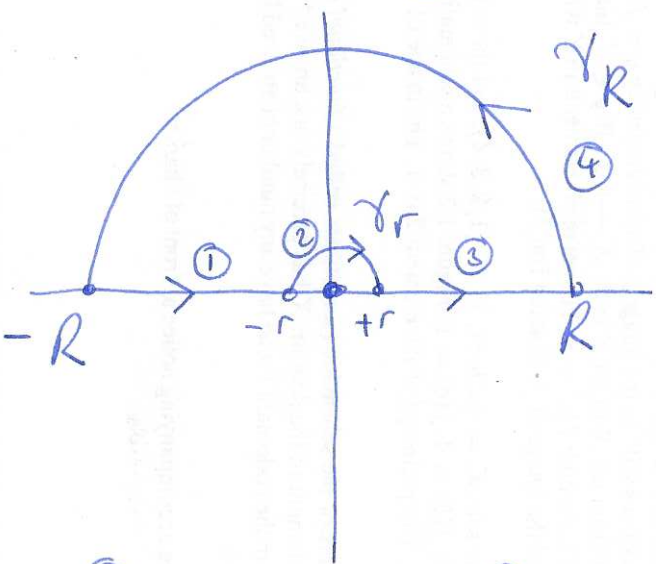
$$\left.\begin{array}{ll} x = 0 & \text{reflection} \\ x \neq 0 & \text{glide reflection} \end{array}\right\}.$$

Example $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ consider $f(z) = \dfrac{e^{iz}}{z}$

pole at 0

$z = x + iy$

C



$f(z)$ is analytic on and in C.

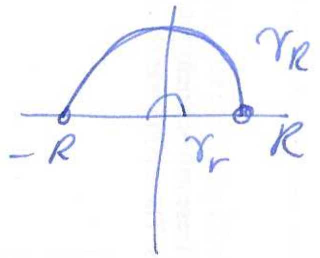
$$\int_C f(z)\,dz = 0.$$

$$\underbrace{\int_{-R}^{-r} \frac{e^{ix}}{x}\,dx}_{1} + \underbrace{\int_{\gamma_r} \frac{e^{iz}}{z}\,dz}_{2} + \underbrace{\int_r^R \frac{e^{ix}}{x}\,dx}_{3} + \underbrace{\int_{\gamma_R} \frac{e^{iz}}{z}\,dz}_{4} = 0$$

take limits: $r \to 0$

$R \to \infty$.

$$\int_{-\infty}^{0} \frac{e^{ix}}{x}\,dx + \lim_{r\to 0}\int_{\gamma_r}\frac{e^{iz}}{z}\,dz + \int_{0}^{\infty} \frac{e^{ix}}{x}\,dx + \underbrace{\lim_{R\to\infty}\int_{\gamma_R}\frac{e^{iz}}{z}\,dz}_{④} = 0$$



④ : $$\int_{\gamma_R}\frac{e^{iz}}{z}\,dz \underset{\text{integrate by parts}}{=} \left.\frac{e^{iz}}{iz}\right|_{-R}^{R} + \int_{\gamma_R}\frac{e^{iz}}{iz^2}\,dz$$

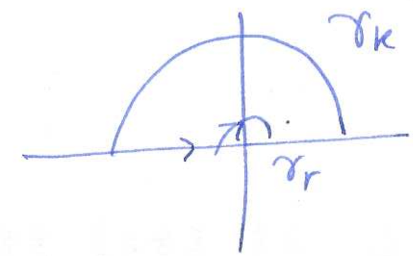
$$= \frac{e^{iR}+e^{-iR}}{iR} + \frac{1}{i}\int_{\gamma_R}\frac{e^{iz}}{z^2}\,dz$$

$2\cos R$

$$\left|\int_{\gamma_R}\frac{e^{iz}}{z}\,dz\right| \leq \left|\frac{e^{iR}+e^{-iR}}{iR}\right| + \left|\int_{\gamma_R}\frac{e^{iz}}{z^2}\,dz\right| \leq \underbrace{\frac{2}{R}+\frac{\pi R}{R^2}}$$

$\longrightarrow 0$ as $R\to\infty$.

① $\lim_{r \to 0} \int_{\gamma_r} \frac{e^{iz}}{z} \, dz$.



$\frac{e^{iz}}{z} \leftarrow$ Laurent expansion at 0

$$\frac{1}{z}\left(1 + iz + \frac{(iz)^2}{2!} + \cdots\right) = \frac{1}{z} + P(z).$$

$$\lim_{r \to 0} \int_{\gamma_r} \frac{e^{iz}}{z} \, dz = \underbrace{\lim_{r \to 0} \int_{\gamma_r} \frac{1}{z} \, dz}_{} + \underbrace{\lim_{r \to 0} \int_{\gamma_r} P(z) \, dz}_{\to 0}$$

$|P(z)| \leq M$ close to 0

$$\int_{\gamma_r} \frac{1}{z} \, dz = \int_{\pi}^{0} \frac{1}{re^{i\theta}} \, ire^{i\theta} \, d\theta$$

$|-| \leq r\pi |P(z)|$.

$\to 0$ as $r \to 0$

$$\int_{\pi}^{0} i \, d\theta = -\pi i$$

parameterization

$c(r) = re^{i\theta}$ $\quad c'(r)\ ire^{i\theta}$.

$$\int_{-\infty}^{0} \frac{e^{ix}}{x}\,dx - \pi i + \int_{0}^{\infty} \frac{e^{ix}}{x}\,dx = 0 .$$

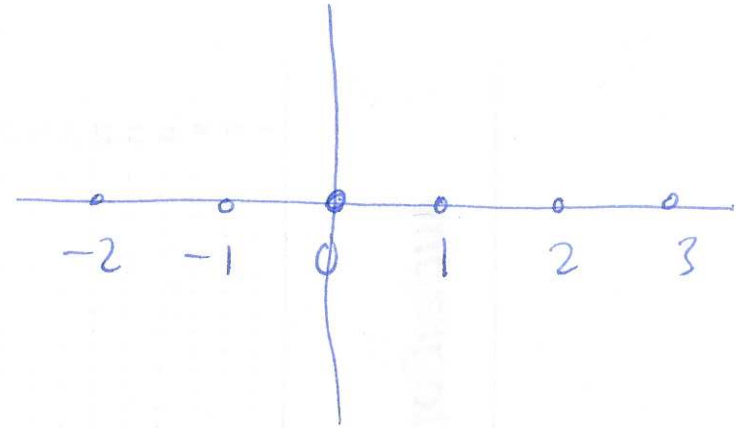
$$2\int_{0}^{\infty} \frac{e^{ix}}{x}\,dx = \pi i . \qquad e^{ix} = \cos x + i \sin y$$

$$2\Big(\underbrace{\int_{0}^{\infty} \frac{\cos x}{x}\,dx}_{0} + \underbrace{i\int_{0}^{\infty} \frac{\sin x}{x}\,dx}\Big) = \pi i .$$

$$\therefore \quad \boxed{\int_{0}^{\infty} \frac{\sin x}{x} = \frac{\pi}{2}}$$

Example $\pi \cot(\pi z) \leftarrow$ simple poles at $\mathbb{Z} \subseteq \mathbb{R}$.

residues:



$$\lim_{z \to n} (z-n)\, \pi \cot(\pi z)$$

$$\lim_{z \to n} \frac{(z-n)\, \pi \cos(\pi z)}{\sin(\pi z)} = (-1)^n \lim_{z \to n} \pi \frac{z-n}{\sin(\pi z)}.$$

$$\overset{l'H}{=} (-1)^n \lim_{z \to n} \frac{\cancel{\pi}}{\cancel{\pi} \cos(\pi z)} = \frac{(-1)^n}{(-1)^n} = 1.$$

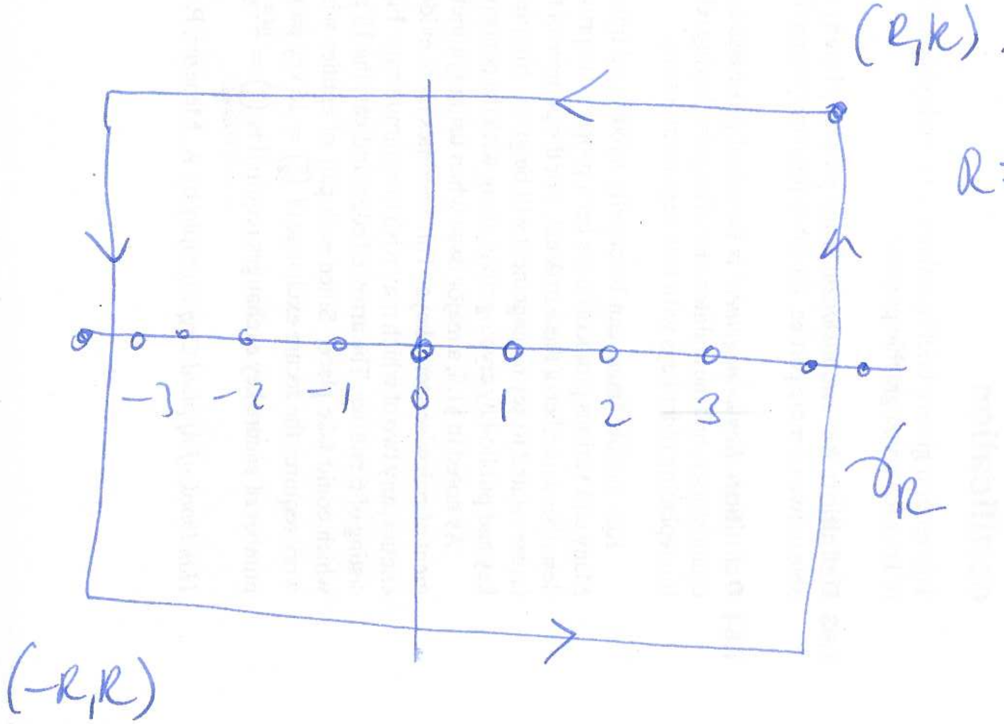
$$f(z) = \frac{\pi \cot(\pi z)}{z^2}.$$

aim $\sum \frac{1}{n^2}$.

$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

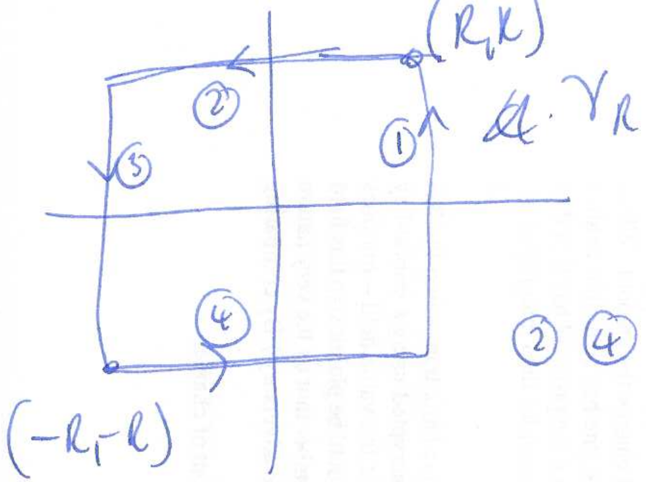
poles at $\mathbb{Z}$. $n \neq 0$ simple pole with residue $\frac{1}{n^2}$.

at 0: pole of order 3. $\operatorname{Res}_0 f(z)$.



$R = N + \frac{1}{2}$.

$$\int_{\gamma_R} f(z)\,dz = \operatorname{Res}_0 f(z) + 2\sum_{n=1}^{N} \frac{1}{n^2}.$$



$$\int_{\gamma_R} \frac{\pi \cot(\pi z)}{z^2}\,dz.$$

② ④ claim $|\cot(z)| \le 2$ if $|\underbrace{\mathrm{Im}(z)}_{y}| \ge 1$.

$$|\cos^2 z| = |\cos(x+iy)|^2 = |\cos(x)\underbrace{\cos(iy)}_{\cosh(y)} - \sin(x)\underbrace{\sin(iy)}_{i\sinh(y)}|^2.$$

$$= \cos^2 x \cosh^2 y + \underbrace{\sin^2 x}_{1-\cos^2 x} \sinh^2 y = \cos^2 x\,(\cosh^2 y - \sinh^2 y) + \sinh^2 y$$

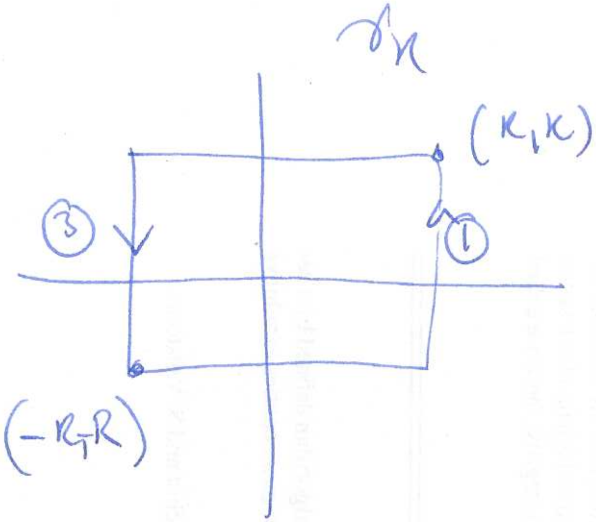
$$= \underbrace{\cos^2 x}_{\le 1} + \sinh^2 y$$

$$|\sin^2 z| = \underbrace{\sin^2 x}_{\le 1} + \sinh^2 y$$

$$|\cot z|^2 \le \frac{1+\sinh^2 y}{\sinh^2 y} \le 2.$$

$$\int_{②} f(z)dz \le \frac{2}{R^2}\cdot R.$$

④ $\to 0$ as $R \to \infty$.



$x = (N+\frac{1}{2})$

$z = x+iy$

$\cot(\pi z) = \dfrac{\cos(\pi z)}{\sin(\pi z)}$ $\qquad \dfrac{\cos(\pi(N+\frac{1}{2})+iy))}{\sin(\pi((N+\frac{1}{2})+iy))}$.

$\left|\dfrac{\cos(\pi iy)}{\sin(\pi iy)}\right| \leq \dfrac{\cosh(\pi y)}{\sinh(\pi y)} \leq 2$. $\qquad \left|\int_{①}^{②}\right| \leq \dfrac{1}{R^2} R$.

$\longrightarrow 0$

as $R \to \infty$.

$$\lim_{R\to\infty} \int_{\gamma_R} \cancel{f(z)} \frac{\pi\cot(\pi z)}{z^2}\, dz = 0.$$

$$2\sum_{n=1}^{\infty} \frac{1}{n^2} = \operatorname*{Res}_0 f(z).$$

$$f(z) = \frac{\pi\cot(\pi z)}{z^2} \qquad ① \quad \lim_{z\to 0} \frac{d^2}{dz}\,\frac{1}{2}\, z^3\, \frac{\pi\cot(\pi z)}{z^2}$$

$$② \quad \pi\,\frac{\cos(\pi z)}{z^2\sin(\pi z)} = \frac{\pi\left(1-\frac{(\pi z)^2}{2!}+\frac{(\pi z)^4}{4!}-\cdots\right)}{z^2\left(\pi z-\frac{(\pi z)^3}{3!}+\frac{(\pi z)^5}{5!}-\cdots\right)}$$

$$\frac{1}{z^3}\,\frac{\left(1-\frac{(\pi z)^2}{2!}+\frac{(\pi z)^4}{4!}\cdots\right)}{\left(1-\underbrace{\frac{\pi^2 z^2}{3!}+\frac{\pi^4 z^4}{5!}+\cdots}_{w}\right)} \qquad \frac{1}{1-w} = 1+w+w^2+\cdots$$

$$\frac{1}{z^3}\left(1-\frac{(\pi z)^2}{2!}+\frac{(\pi z)^4}{4!}\cdots\right)\left(1+\left(\frac{\pi^2 z^2}{3!}-\frac{\pi^4 z^4}{5!}+\cdots\right)+(-)^2+\cdots\right)$$

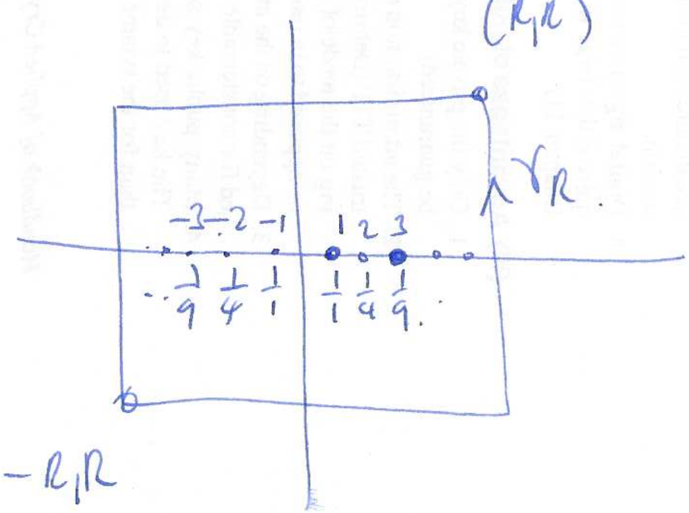
$$\frac{1}{z^3}\left(\cdots-\frac{\pi^2 z^2}{2}+\frac{\pi^2 z^2}{6}\right) = \frac{1}{z}\pi^2\left(\frac{1}{6}-\frac{1}{2}\right) = -\frac{1}{\cancel{z}}\,\frac{\pi^2}{3}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{1}{2} \operatorname{Res}_0 f(z) = -\frac{1}{2} \cdot -\frac{\pi^2}{3} = \frac{\pi^2}{6}.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

similar arguments work for $\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$

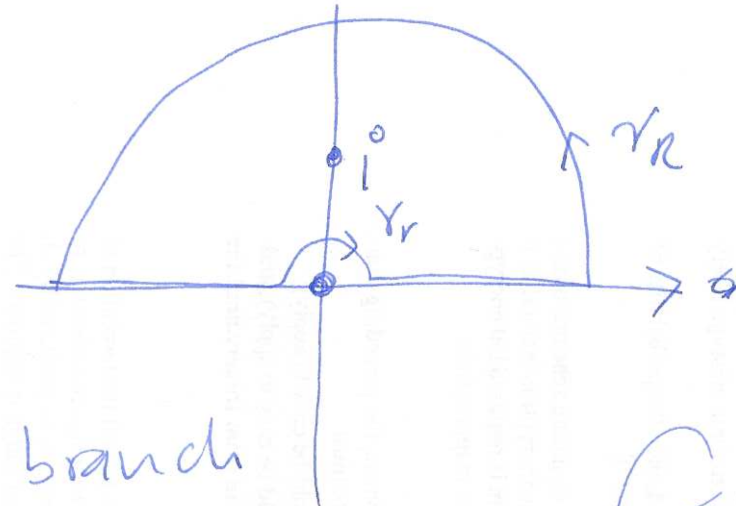
$$\sum_{n=1}^{\infty} \frac{1}{n^3} = ?.$$



$$\underbrace{\int_{\gamma_R} f(z)\,dz}_{0} = \left(\operatorname{Res}_0 f(z) + 2\sum_{n=1}^{N} \frac{1}{n^2}\right) \times 2\pi i.$$

$$\sum_{n=1}^{N} \frac{1}{n^2} = -\frac{1}{2} \operatorname{Res}_0 f(z).$$

Example. $$\int_0^\infty \frac{\ln x}{(x^2+1)^2}\,dx$$



$$f(z) = \frac{\ln(z)}{(z^2+1)^2}$$

choose a branch

$$-\pi < \arg(z) < \pi.$$

$$\int_{\gamma_R} f(z)\,dz = 2\pi i \operatorname*{Res}_{i} f(z).$$