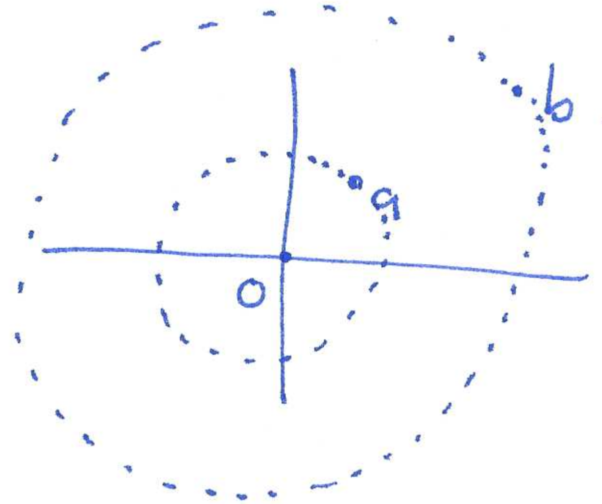
11 Q2 c)

$$f(z) = \frac{1}{(z-a)(z-b)}$$



a)

$$\frac{A}{z-a} + \frac{B}{z-b} = \frac{A(z-b) + B(z-a)}{(z-a)(z-b)}$$

$$= \underline{z(A+B) - aB - bA}$$

$A = -B$

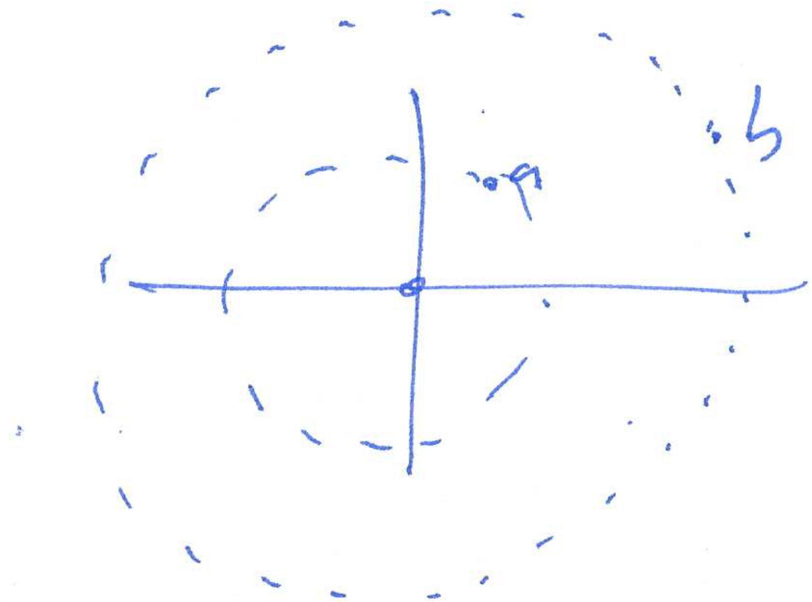
$\cancel{(a+b)A = 1}$

$aA - bA = 1$

$A(a-b) = 1$ $\qquad A = \frac{1}{a-b}$

$$\frac{1/_{a-b}}{z-a} + \frac{1/_{b-a}}{z-b}$$

$$\frac{1/(a-b)}{z-a} + \frac{1/(b-a)}{z-b}. \qquad \frac{1}{1-z} = 1+z+z^2+\cdots$$



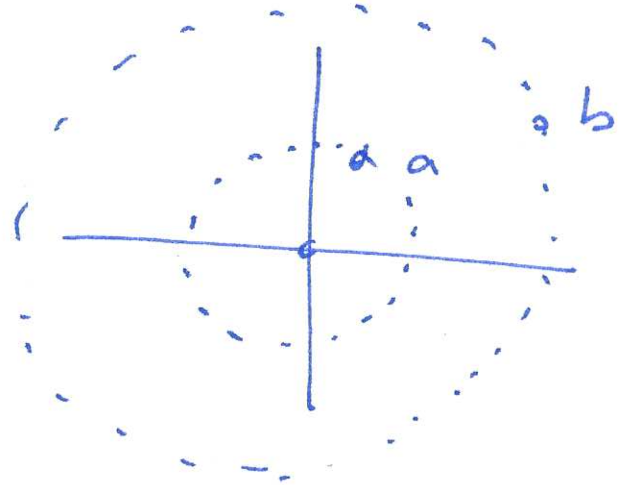
$$\frac{1/(b-a)}{a-z} + \frac{1/(a-b)}{b-z}$$

$$\frac{1/a(b-a)}{1-z/a} + \frac{1/b(a-b)}{1-z/b}.$$

$$\frac{1}{a(b-a)}\left(1+\frac{z}{a}+\frac{z^2}{a^2}+\cdots\right) + \underbrace{\frac{1}{b(a-b)}\left(1+\frac{z}{b}+\frac{z^2}{b^2}+\cdots\right)}.$$

radius of convergence $\min\{|a|,|b|\}$.

$= |a|$.


a
a
b.

recall

isolated singularities

- removable ✓.
- poles ✓.
- essential sing.

pole of order $m > 1$

Laurent expansion is

$$f(z) = \frac{c_{-m}}{(z-z_0)^m} + \cdots + \frac{c_{-1}}{z-z_0} + c_0 + c_1(z-z_0) + \cdots$$

$$(z-z_0)^m f(z) = c_{-m} + \cdots + c_{-1}(z-z_0)^{m-1} + c_0(z-z_0)^m + \cdots$$

differentiate $(m-1)$ times.

$$\frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z) \right) = (m-1)!\, c_{-1} + \frac{m!}{1!} c_0 (z-z_0) + \cdots$$

$$(m-1)!\, c_{-1} = \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z) \right).$$

$$c_{-1} = \operatorname*{Res}_{z_0} f(z) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z) \right)$$

agrees w/ pole $m=1$. case.

<u>Example</u>

$$f(z) = \frac{e^z}{(z-1)^n} \quad \leftarrow \text{pole of order } n \text{ at } z = 1.$$

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z \to 1} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left((z-1)^n \frac{e^z}{(z-1)^n} \right).$$

$$= \lim_{z \to 1} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left(e^z \right)$$

$$= \lim_{z \to 1} \frac{e^z}{(n-1)!} = \frac{e}{(n-1)!}$$

§12 Residues

Logarthmic residues and arguments.

<u>Defn</u> the <u>logarithmic residue</u> of $f(z)$ at a is $\underset{z=a}{\mathrm{Res}} \dfrac{f'(z)}{f(z)}$

this is the residue of $\dfrac{d}{dz}\left(\ln(f(z))\right)$

$$= \frac{1}{f(z)} \cdot f'(z).$$

Example

$$f(z) = z^n$$

$$f'(z) = n z^{n-1}.$$

$$\boxed{\ln(f(z))}$$

$$\frac{f'(z)}{f(z)} = \frac{n z^{n-1}}{z^n} = \frac{n}{z}. \qquad c_{-1} = n$$

$$\operatorname*{Res}_{z=0} \frac{f'(z)}{f(z)} \text{ when } f(z) = z^n \text{ is } n.$$

awkward : $\frac{1}{z^2}$ $n=-2$ pole of order 2.

z^2 $n=2$ zero of order 2.

• suppose a is a zero of order $\alpha \in \mathbb{N}$.

$$f(z) = \underset{\substack{\neq \\ 0}}{c_\alpha} (z-a)^{\alpha} + c_{\alpha+1} (z-a)^{\alpha+1} + \cdots$$

$$f'(z) = \alpha c_\alpha (z-a)^{\alpha-1} + \cdots$$

$$\frac{f'(z)}{f(z)} = \frac{\alpha c_\alpha (z-a)^{\alpha-1} + \cdots}{c_\alpha (z-a)^{\alpha} + \cdots}$$

$$\frac{1}{z-\cancel{\alpha}\,a}\left[\frac{\alpha + (\alpha+1)\frac{c_{\alpha+1}}{c_\alpha}(z-a) + \cdots}{1 + \frac{c_{\alpha+1}}{c_\alpha}(z-a) + \cdots}\right].$$

$$\frac{f'(z)}{f(z)} = \frac{\alpha}{z-a} + c_0' + c_1' z + \cdots$$

$$\operatorname*{Res}_{z=a} \frac{f'(z)}{f(z)} = \alpha = c_{-1}$$ i.e. the order of the zero.

- Let b be a pole of order β.

$$f(z) = \frac{\overset{\neq 0}{c_{-\beta}}}{(z-b)^{\beta}} + \frac{c_{-\beta+1}}{(z-b)^{\beta-1}} + \cdots$$

$$f'(z) = \frac{-\beta\, c_{-\beta}}{(z-b)^{\beta-1}} - \frac{(\beta-1)\, c_{-\beta+1}}{(z-b)^{\beta-2}} - \cdots$$

$$\frac{f'(z)}{f(z)} = \frac{\dfrac{-\beta c_{-\beta}}{(z-b)^{\beta-1}} - \dfrac{(\beta-1)\, c_{-\beta+1}}{(z-b)^{\beta-2}} + \cdots}{\dfrac{c_{-\beta}}{(z-b)^{\beta}} + \dfrac{c_{-\beta+1}}{(z-b)^{\beta-1}} + \cdots}$$

$$= \frac{1}{z-b}\left[\frac{-\beta c_{-\beta} - (\beta-1) c_{-\beta+1}(z-b) \cdots}{c_{-\beta} + c_{-\beta+1}(z-b) + \cdots}\right].$$

$$= \frac{-\beta}{z-b} + c_0' + c_1'(z-b) + \cdots$$

$$\operatorname*{Res}_{z=b} \frac{f'(z)}{f(z)} = -\beta.$$

<u>Thm</u> C piecewise smooth Jordan curve

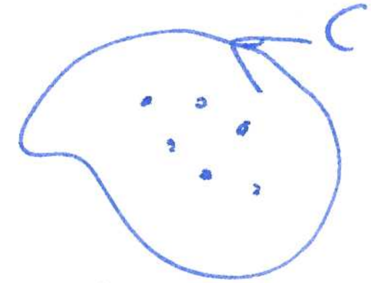
$f(z)$ is analytic in and on C except for poles $b_1, \ldots, b_n$, and $f(z)$ has zeros $a_1, \ldots, a_m$ inside C but not on C. Then

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\, dz = \sum_{k=1}^{m} \alpha_k - \sum_{k=1}^{n} \beta_k$$

$\alpha_k =$ order of the zero a_k

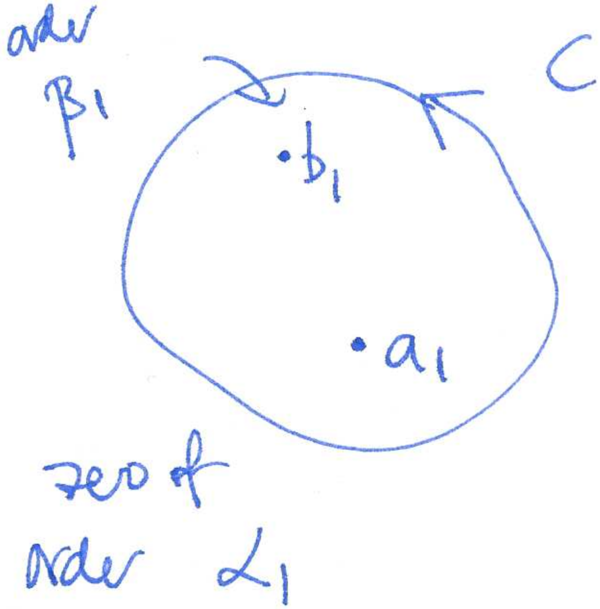
$\beta_k =$ order of the pole b_k.

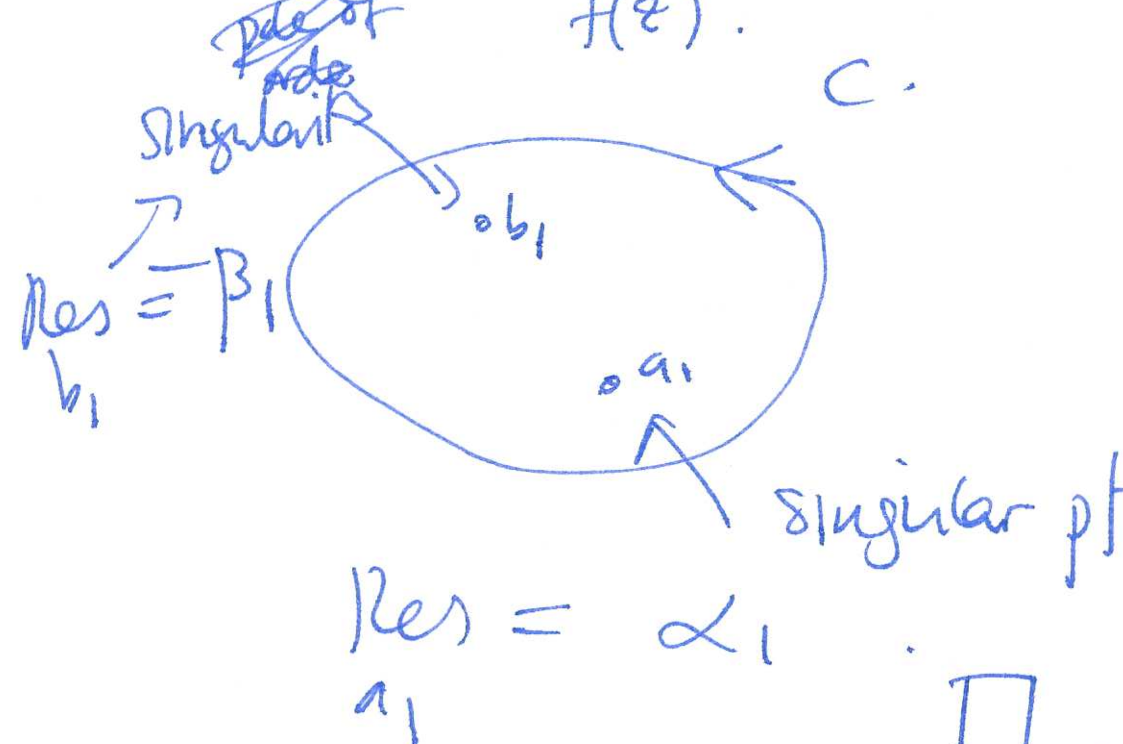
Proof singular points of $\frac{f'(z)}{f(z)}$ are exactly the poles and zeros of $f(z)$.



now apply residue thm.

old $f(z)$ new $\frac{f'(z)}{f(z)}$.





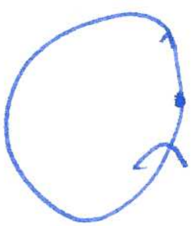
$\square$

Let

N = # of zeros } count with multiplicity
P = # of poles }

$$N - P = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_C \frac{d}{dz}\left(\ln(f(z))\right)dz$$

$$= \frac{1}{2\pi i}\int_C d\ln(f(z)) = \frac{1}{2\pi i}\,\Delta_C \ln(f(z))$$



C

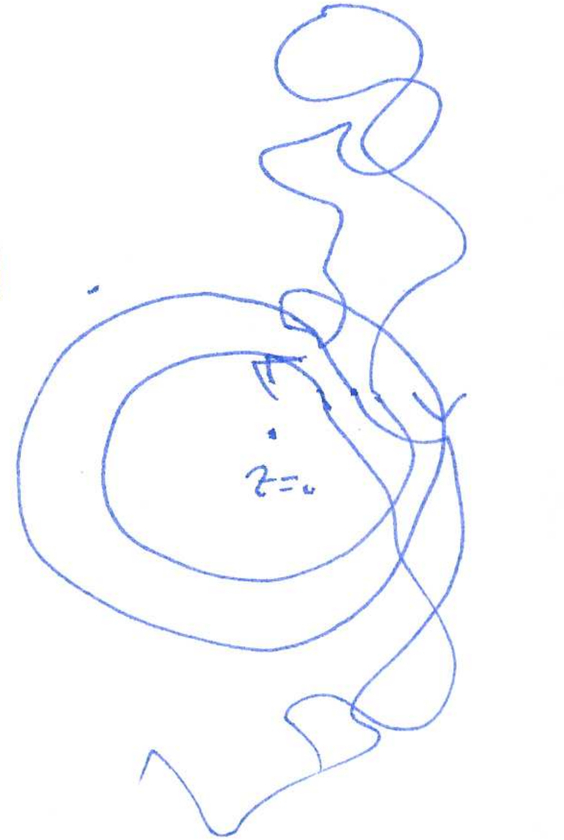
change in $\ln(f(z))$ as z goes round C.

$$\ln(f(z)) = \ln|f(z)| + i\arg(f(z)) + 2\pi i n.$$

$\Delta_C \ln f(z)$

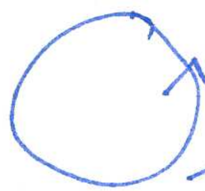
$$\Delta_C \ln(f(z)) = i\,\Delta_C \arg(f(z))$$

$$N - P = \frac{1}{2\pi}\,\Delta_C \arg(f(z))$$



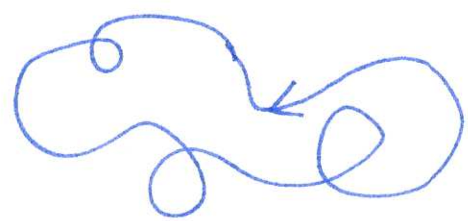
$$f : \mathbb{C} \longrightarrow \mathbb{C}.$$

$f(C)$



C

Jordan curve.

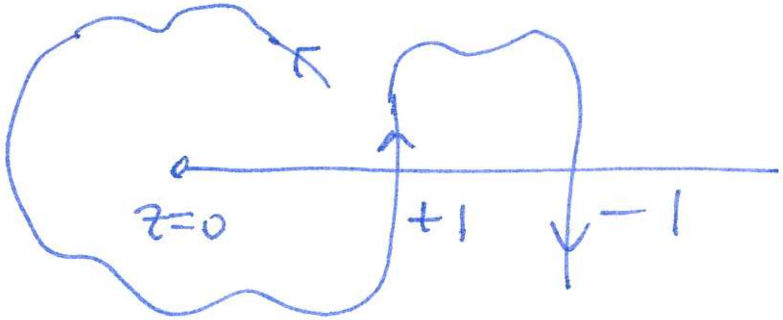


winding number $(z=0)$

+ve direction

n_+ = number of times you go round anti-clockwise

n_- = number of times you go round in −ve direction (clockwise)



$+1$ cross $\mathbb{R}_+$ from bottom to top

-1 cross $\mathbb{R}_+$ from top to bottom

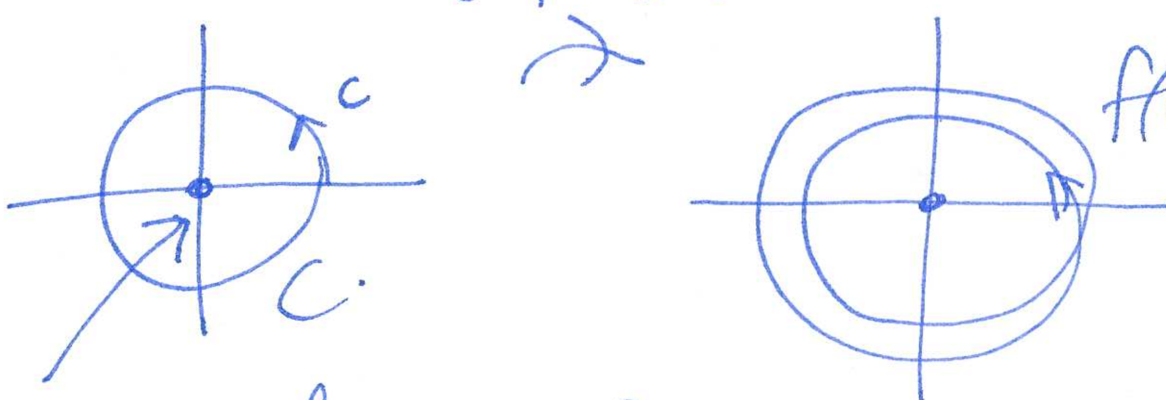
winding number $\boxed{v = n_+ - n_-}$ (Image ~~in~~ of $f(\mathbb{C})$)

so $\Delta_C \arg f(z) = 2\pi v$

$$\boxed{N - P = v}$$

Example $z \mapsto z^2$.

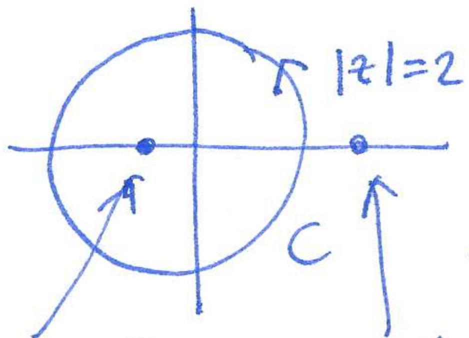
$z \mapsto z^2$



$f(C) \leftarrow$ winding number 2.

zero of order 2.

$|z| = 2$



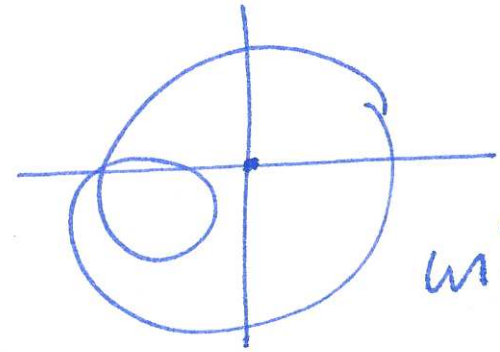
zero of order 1

zero of order 1

$\rightarrow$

$z \mapsto z^2 - 2z - 3$.

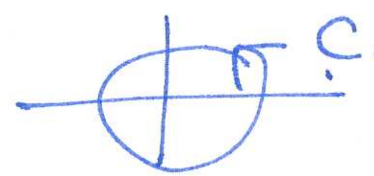
$(z-3)(z+1)$



winding number 1

<u>Thm</u> (Rouché) f, g are analytic in and on C piecewise smooth Jordan curve and suppose

$$|f(z)| > |g(z)| \text{ on } C$$



then $f(z)$ and $f(z) + g(z)$ have the same number of zeros in C.

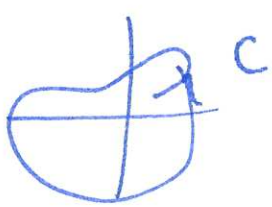
<u>Proof</u> $f(z) \neq 0$ on C $\qquad |f(z)| > |g(z)|$ on C.

$$\Delta_C \arg\left(f(z)+g(z)\right) = \Delta_C \arg\left(f(z)\left(1+\frac{g(z)}{f(z)}\right)\right).$$

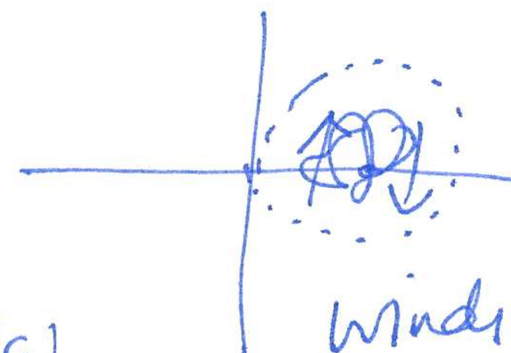
$$= \Delta_C \arg\left(f(z)\right) + \Delta_C \arg\left(1+\frac{g(z)}{f(z)}\right).$$

but $\left|\dfrac{g(z)}{f(z)}\right| < 1$ on C so $1+\dfrac{g(z)}{f(z)}$

stays inside disc $|w-1| < 1$



$\longrightarrow$ $1+\dfrac{g(c)}{f(c)}$



winding #

$V = 0$.

$$\Delta_c \arg\left(1 + \frac{f(z)}{g(z)}\right) = 0.$$

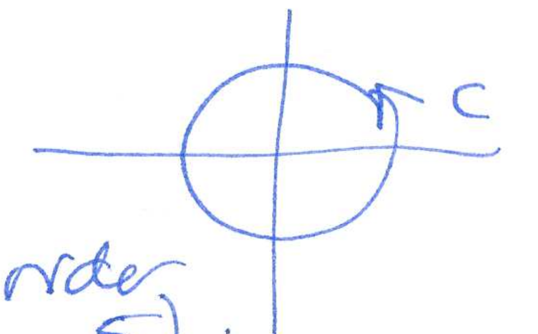
$$\Delta_c \arg\big(f(z) + g(z)\big) = \Delta_c \arg\big(f(z)\big) \quad \square.$$

Example Q: how many zeros does

$\boxed{z^8 - 4z^5 + z^2 - 1}$ (*) have inside the unit circle?

$f(z) = -4z^5 \leftarrow$ 5 zeros inside unit circle.

($z = 0$ is a zero of order 5).



$g(z) = z^8 + z^2 - 1$

$|z| = 1$

$|f(z)| > |g(z)|$ on C

so (*) has 5 zeros in the unit circle..

Thm (Fundamental theorem of algebra)

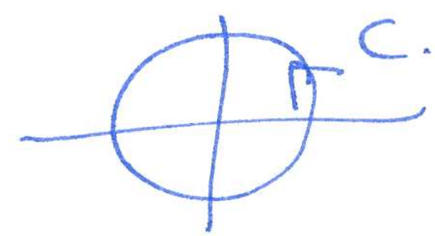
Every polynomial $P(z) = a_0 + a_1 z + \cdots + a_n z^n$, $a_n \neq 0$,

has precisely n zeros $(n \geq 1)$ counted with multiplicity.

Proof set $f(z) = a_n z^n$

$$g(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}$$

consider circle of radius R $\quad |z| = R$

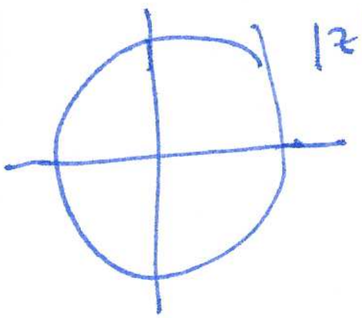


$$|f(z)| = |a_n| R^n$$

$$|g(z)| \leq |a_0| + |a_1| R + \cdots + |a_{n-1}| R^{n-1}$$

$$\lim_{\substack{\cancel{R\to\infty} \\ |z|\to\infty}} \left|\frac{f(z)}{g(z)}\right| \geq \lim_{\substack{\cancel{R\to\infty} \\ R\to\infty}} \frac{|a_n|R^n}{|a_0| + \cdots \quad |a_{n-1}|R^{n-1}} = \infty.$$

so $|f(z)| > |g(z)|$ for all R sufficiently large



apply Rouche. # zeo of $P(z)$

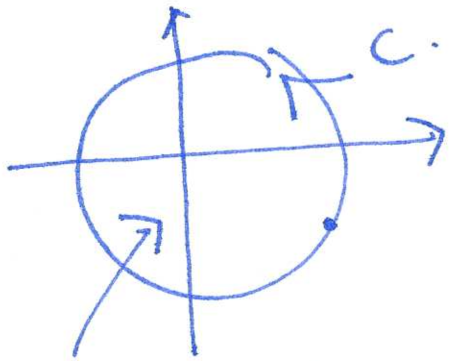
$= \#$ zeros of $a_n z^n$

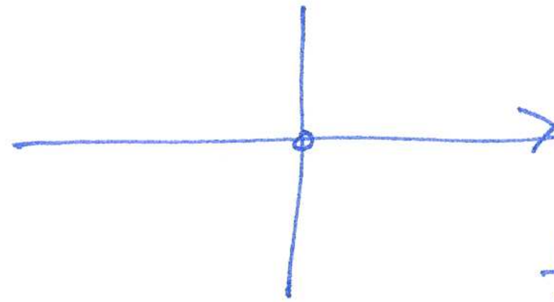
$= n.$ $\square$.

polynomial

of deg n. $\quad a_0 + a_1 z + a_2 z^2 + \cdots + \underset{\substack{\neq \\ 0}}{a_n} z^n$

$z \longmapsto f(z)$





$f(C)$.

poles and zeros inside

not on C

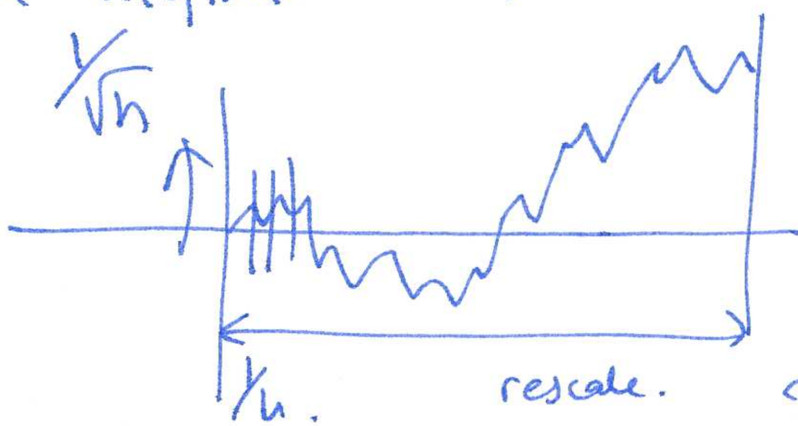
winding number not defined if you hit $\underline{z=0}$.

Brownian motion.

paths cts
but not differentiable.

1d

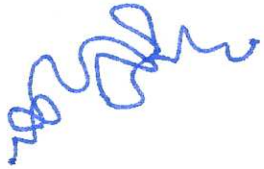
$1/\sqrt{n}$



H T H H T H T T . . . T

$+1\ -1\ +1\ +1\ -1\ +1\ -1\ -1$

length n.

 ← continuous not differentiable & not piecewise smooth

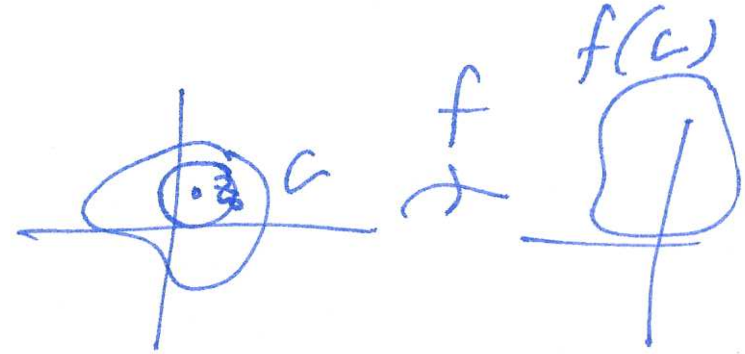
$\mathbb{C}$.

Q: what is the winding number?

Q: how often do you hit zero

about some pt close to zero?

<u>Thm</u> If $f(z)$ is univalent in a domain G then $f'(z) \neq 0$ in G.



<u>Proof</u> pick $z_0 \in G$

spose $f'(z_0) = 0$

first nonzero term after c_1.

$$f(z) = c_0 + \underset{\overset{\shortparallel}{0}}{c_1}(z-z_0) + \cdots + \underset{\overset{\neq}{0}}{c_k}(z-z_0)^k + c_{k+1}(z-z_0)^{k+1} \qquad k \geq 2.$$

converges for $|z-z_0| < r$ inside G.

choose a disc small enough s.t. $f'(z)$ does not vanish on $0 < |z-z_0| < r$

let $g(z) = c_k + c_{k+1}(z-z_0)^{k+1} + \cdots$

doesn't vanish on $0 < |z-z_0| < 1$.

$g(z_0) = c_k$

let $\mu = \min\limits_{|z-z_0|=r} |c_k(z-z_0)^k + c_{k+1}(z-z_0)^{k+1} + \cdots|$.

$a \neq 0 \in \mathbb{C}$ s.t. $|a| < \mu$. Then $f(z) - (c_0 + a)$

$= -a + c_k(z-z_0)^k + c_{k+1}(z-z_0)^{k+1} + \cdots$

has same number of zeos as $\leftarrow k$-zeros

$c_k(z-z_0)^k + c_{k+1}(z-z_0)^{k+1} + \cdots = (z-z_0)^k(c_k + c_{k+1}(z-z_0)^1 + \cdots)$

$(n \geq 2)$ each zero is simple

$$(f(z) - (c_0 + a))' = f'(z) \neq 0.$$

on $0 < |z - z_0| < r$

$\Rightarrow$ there are two zero $\#$ f is univalent. $\square$.

<u>Corollary</u> If $f(z)$ univalent

then $f(z)$ is conformal.